This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Educ T
168.34
725

Educ T 168.34.795

HARVARD COLLEGE
LIBRARY

THE ESSEX INSTITUTE
TEXT-BOOK COLLECTION

GIFT OF
GEORGE ARTHUR PLIMPTON
OF NEW YORK

JANUARY 25, 1924

3 2044 097 047 013

J. P. Feſsenden

Bowd Coll

2d Term Soph Year

ELEMENTS

OF

PLANE TRIGONOMETRY.

BY WILLIAM SMYTH, A. M.

PROFESSOR OF MATHEMATICS IN BOWDOIN COLLEGE.

BOSTON.
LILLY, WAIT, COLMAN, AND HOLDEN.
1834.

Educ T 168.34.795

HARVARD COLLEGE LIBRARY
GIFT OF
GEORGE ARTHUR PLIMPTON
JANUARY 25, 1924

Entered, according to act of Congress, in the year 1834, by WILLIAM SMYTH, A. M. in the Clerk's Office of the District of Maine.

PRESS OF J. GRIFFIN, BRUNSWICK, ME.

ADVERTISEMENT.

In the following treatise the writer has endeavored to present the principles of plane trigonometry in a natural and connected order, to show the application of each principle as it is introduced, and in general to lead the student to feel the want of a new principle before proceeding to its investigation. The work has been prepared chiefly with reference to a college class. Regard has nevertheless been had to the wants of students in our public seminaries generally, in which trigonometry and its applications are taught. Several of the more recent treatises on trigonometry have been consulted in the course of the work; the most assistance however has been derived from those of Lacroix and Bezout.

The Geometry referred to is that of Legendre in the Cambridge Mathematics.

Bowdoin College, Jan. 1834.

TABLE OF CONTENTS.

SECTION. PAGE.

I. Preliminary Remarks 5

II. Of the Solution of Triangles by Geometrical Construction . 6

III. Of the Solution of Triangles by Arithmetical Calculation . . 12

IV. Of the method of forming a Series of Triangles, having angles of every possible magnitude 15

V. Of the Method of calculating the sides in the Series of Triangles constructed in the Quadrant of a Circle 17

VI. Of the Solution of Triangles by the Table of Sines and Cosines 24

VII. Of the second Method of constructing and calculating a Series of Triangles, having angles of every possible value . . 25

VIII. Of the Solution of Triangles by the Table of Tangents and Secants 28

IX. Of Cotangents and Cosecants 30

X. Miscellaneous Remarks on the Trigonometrical Tables . . 31

XI. Of the Solution of oblique angled Triangles 33

XII. Mensuration of Heights and distances 40

XIII. Of the Line of natural Sines, Tangents, &c. 50

XIV. Of the Line of logarithmic Sines, Tangents, &c. 51

XV. Trigonometrical Analysis 55

ELEMENTS

OF

PLANE TRIGONOMETRY.

SECTION I.

Preliminary Remarks.

1. The triangle ABC (fig. 1), bounded by right lines, and having in consequence all its parts situated in the same plane, is called a *plane triangle*.

2. The plane triangle DEF (fig. 2), in which one of the sides DE forms with another EF a right angle, is called a *right angled triangle*. Any other triangle ABC (fig. 1), in which neither of the angles is a right angle, is called an *oblique triangle*.

3. In the right angled triangle DEF (fig. 2), the part DF opposite the right angle, is called the *hypothenuse* ; the other two parts DE, EF are called the *sides* ; sometimes also *the perpendicular and base*.

4. In a plane triangle there are six parts to be considered, viz, three angles and three sides. In order to calculate the rest certain of these parts must be given.

5. Let the lines a and b (fig. 3) be two of the sides of a proposed triangle. These lines may evidently make with each other any angle whatever ; of consequence they may

be sides in an indefinite number of triangles ABC, ABC′, &c. The remaining parts of a triangle cannot therefore be determined, when two of the sides are the only parts, which are given. Again let one of the sides AB and an angle A (fig. 4) be two known parts in a proposed triangle ; it is evident, that these alone will not be sufficient to determine the rest ; for the side AB and the angle A may also belong to an indefinite number of triangles ABC, ABC′, &c. Moreover if two of the angles, or indeed if all the angles, are the only parts, which are given, the remaining parts must still be indeterminate ; for in the triangle ABC (fig. 5,) if the lines B′ C′, B″ C″, &c. be drawn parallel to BC, an indefinite number of triangles may be formed, in each of which the angles will be respectively equal, while the sides are different in value. *In order then, that a triangle may be calculated, three at least of its parts should be given, one of which must be a side.*

6. That part of trigonometry, which relates to the calculation of plane triangles, is called *Plane Trigonometry.*

SECTION II.

Of the solution of Triangles by Geometrical construction.

7. In order to calculate a triangle geometrically, its different parts must be measured ; among these parts there are two kinds of quantities, viz, angles and sides.

8. Since the sides of plane triangles are right lines, their values are expressed in the usual measures of extension, as feet, yards, &c. To represent these measures for the purpose of calculating a triangle geometrically, any line AB (fig. 6) may be divided into a convenient number of equal parts ; one of these being considered as unity may be further divided into equal portions for fractional parts of unity. A line divided in this manner is called a *scale of equal parts.*

In the example figure 6, the line AB is divided into eleven equal portions ; of these the extreme one on the left is divided into ten equal parts, the remainder being numbered 1, 2, 3, &c. The principal divisions of this scale may be considered as feet, miles, &c.; then the subdivisions will be tenths of a foot, mile, &c. The principal divisions may also be regarded as ten feet, ten miles, &c.; in this case the subdivisions will represent feet, miles, &c.

A convenient construction for a scale of equal parts is represented figure 7. It consists, when intended for a decimal scale, of eleven lines drawn parallel to each other at equal distances, and divided into convenient portions by perpendicular lines. In the extreme division on the left, the upper and lower lines are divided into ten equal parts, and the subdividing points are connected by *diagonal* lines, that is, by lines drawn from the first point on the lower line to the second on the upper. Then by similar triangles $ac : ab :: cd : bo$, that is, cd is one tenth of the subdivisions, or one hundredth of the primary divisions of the scale. In like manner ef may be shown to be two hundredths, and gh three hundredths of $a1$.

Let it be required, for example, to take off the number 255 from this scale. Considering the subdivisions as containing ten each, we extend the compasses from figure 2 of the principal divisions to 5 of the subdivisions for 250 ; then by opening the compasses to the corresponding extent on the fifth of the parallel lines, we shall obtain the length required.

9. In order to estimate the magnitude of the angles, it would seem more natural to refer them to the right angle, as the unit of measure ; since, in this case, the quantities to be measured would be the same in kind with the quantity assumed as their measure. It is found, however, more convenient in practice to measure angles by arcs of circles. In

the circle AC (fig. 8), in whatever ratio the angle DBC at the centre increases or diminishes, the arc CD, on which it stands, increases or diminishes in the same ratio (Geom. 122); the arc CD may therefore be assumed, as the measure of this angle. In like manner any other concentric arc EF, intercepted by the sides of the angle DBC, may be used as its measure. The arc CD, indeed, considered as a magnitude of a certain length, is manifestly greater than the arc EF. In the measure of angles, however, it is not the absolute value of the arcs, which we regard, but merely their ratio to an entire circumference. Since then the arc EF is to the circumference EG, as the arc CD to the circumference CA ; considered as parts of the entire circumferences, to which they respectively belong, the arcs CD and EF are equal in quantity. *The measure of an angle is then the arc of a circle, having its centre in the vertex of the angle, and intercepted by the lines, which form its sides.* In the measure of an angle it is therefore immaterial, what may be the size of the circle employed, provided it cut the sides of the angle. In comparing different angles, however, the arcs, on which they are measured, should be described with equal radii.

10. The entire circumference of every circle, whether great or small, is divided into 360 equal parts called degrees, each degree into 60 equal parts called minutes, each minute into 60 equal parts called seconds, each second into 60 equal parts called thirds, &c. The character used to denote degrees is °, placed at the right hand of the unit figure and a little above it, that for minutes is ′, that for seconds is ″, that for thirds is ‴, so that 40° 30′ 25″ 10‴ signifies 40 degrees, 30 minutes, 25 seconds, and 10 thirds. This division, however, is entirely arbitrary, and any other may be assumed at pleasure. We may, for instance, after the manner of the French, divide the circle into 400 equal parts for

degrees, and each degree into 100 equal parts for minutes, with further subdivisions according to the principles of the decimal progression.

11. We have seen art. 9, that the measure of any angle DBC (fig. 8) is the arc CD of a circle CA ; but the chord CD, being the distance in a right line between the extremities of the arc CD, will determine the magnitude of this arc. If then the several chords for each degree, minute, &c., contained in the arc AB (fig. 9), equal to a quadrant or fourth part of a circle, be transferred to the line BD, the line BD, denominated in this case a *line of chords*, may with convenience be employed in the measurement of angles.

Let it be required for example, to ascertain the magnitude of the angle ABC (fig. 10) by the line of chords. With the extent from 0 to 60 on the line of chords, we describe from B as a centre, an arc cutting the two lines, which include the angle in the points *m* and *n* ; then since the chord of 60° is equal to radius (Geom. 271), this arc will have the same radius with that, to which the given scale of chords belongs ; consequently, the distance between the points of intersection *m* and *n*, applied to the line of chords, will give the measure of the angle in degrees.

Conversely, let it be required to make an angle ABC of a given magnitude, 30° for example. Having drawn a line BC, from the point B with a radius B*n*, equal to the chord of 60° on the line of chords, describe an arc *n m* ; the extent from B to 30° on the line of chords applied from *n* on the arc *nm* will give the point, through which if the line BA be drawn, the angle ABC will be formed of the required magnitude.

12. Let there now be given two of the sides of a triangle and the angle contained between them, to find, by construction, the remaining sides and angles. From any convenient

scale of equal parts we take the line AB (fig. 1), equal to one of the given sides; at the point A, one of the extremities of this line, we make the angle A equal to the given angle, and draw through this the line AC equal to the other given side; we then connect the points B and C by the line BC; the triangle being thus constructed, the side BC, and the remaining angles may be measured.

In general to calculate a triangle geometrically, we construct the triangle by means of the parts, which are given; the sides are laid down from a scale of equal parts, and the angles from a line of chords; the required parts are then measured, the sides upon the same scale of equal parts, and the angles upon the same line of chords, from which the given parts were laid down.

The following examples exhibit the different cases, which occur in the solution of triangles.

13. PROB. 1. *The three sides of a triangle* being given, to find the angles.

Ex. 1. In the triangle ABC (fig. 1), given the side AB 39 rods, BC 45, and AC 38 rods; to find the angles.

Ex. 2. Given the three sides 53.5, 82.08, and 46.7; to find the angles.

14. PROB. 2. *Two sides and the included angle* being given, to find the other side and angles.

Ex. 1. In the triangle ABC (fig. 11), given the side AB 60 rods, the side BC 100 rods, and the angle B 45° 30′; required the remaining parts.

Ex. 2. Given two of the sides of a triangle equal to 345 and 759 feet respectively, and the angle contained between them 53° 45′, to find the remaining parts.

15. PROB. 3. *Two sides and an angle opposite* one of them being given, to find the remaining side and the other two angles.

Ex. 1. Given in the triangle ABC (fig. 12) the side AB

57.5 feet, the side BC 103 feet, and the angle A 99° 15′; required the third side and the two other angles.

Ex 2. Given in the triangle ABC (fig.13) the side AB 72 feet, the side AC 41.5 feet, and the angle B 30° ; required the remaining parts.

16. In this last example, in which the side opposite the given angle is less, than the other given side, the solution is *ambiguous ;* for if from the point A, with a radius equal to AC, we describe the arc CC′ cutting the line BC in C′, and draw AC′, a second triangle ABC′ will be formed, which will also contain the three given parts. To determine then, which of these triangles is intended in the question proposed, we must know, in addition to the given parts, whether the angle C be acute or obtuse. With respect to this case, we may remark in passing, that the angles AC′B, ACB are together equal to two right angles. Indeed the angles AC′B, AC′C are together equal to two right angles, but by construction the triangle AC′C is isosceles, the angle ACB is therefore equal to AC′C, whence AC′B, ACB are together equal to two right angles. The angles ACB AC′B are therefore, the supplements of each other, since we understand by the *supplement of an angle that which must be added to this angle in order to make two right angles.*

17. Prob. 4. *One side and two of the angles* being given, to find the remaining sides.

Ex. 1. Given the side AB of a triangle (fig. 14), 40 feet, the angle ABC 52°, and the angle BAC 70°; to find the remaining sides.

Ex. 2. Given one side of a triangle 600 yards, the angle opposite 65°, one of the adjacent angles 40° 30′ ; required the remaining parts.

18. It is now evident, that we may always construct a triangle, when three of its parts are given, provided that one

of them be a side. If, however, the given parts be two of the sides and an angle opposite one of them, and the side opposite the given angle be less, than the other given side, to determine the triangle , we must know whether the angle sought be acute or obtuse.

19. From the preceding examples it is easy to see, that the solution, obtained by geometrical construction, will not on account of the imperfection of instruments, be precisely the truth, but only an approximation to it. It is then desirable to ascertain methods of calculating triangles by numerical operations, since by means of these the calculation may be carried to any degree of exactness at pleasure.

SECTION III.

Of the solution of Triangles by Arithmetical calculation.

20. In the solution of a triangle, it is obvious, that the magnitude of the angles will always come into consideration. Thus in the triangle ABC (fig. 1), if it be required to determine the third side AC, the angle at B and the sides AB,BC being given, since the value of AC evidently depends in part upon the magnitude of the angle at B, it is manifest, that the magnitude of this angle must be made to enter into the computation. But the angle at B is measured on the arc of a circle, while the sides are right lines ; of consequence the parts, which are necessarily the elements of the calculation, are different in kind and do not admit of comparison with each other. In endeavoring then, by means of numerical operations, to determine the different parts of a triangle, we are embarrassed at the outset by the difficulty of introducing into the calculation the magnitude of the angles.

21. To show the manner, in which this difficulty may be removed, we suppose that the triangles ABC, *abc* (fig. 15)

are similar, and that the sides *ab*, *bc*, *ac*, of the latter are known; if then in the triangle ABC the side BC be given, it is evident from the nature of similar triangles, that the remaining parts of this last may be determined from those of the other by means of a simple proportion.

Let it then be supposed, that we have determined, by geometrical construction, the ten triangles in Plate 2, the first having the angle at B 10°, the second 15°, the third 20°, and so on; and, for the sake of simplicity, let these triangles each be right angled at C. If it now be proposed to solve the triangle *abc* (fig. 16), right angled at *c*, the side *bc* 73 feet, and the angle *b* 40°, it is evident, on inspecting the series of computed triangles, that the triangle No. 7 is similar to the triangle proposed; since these triangles are both right angled, and one of the acute angles in the one is equal to an acute angle in the other. By means then of the computed triangle in the series, the remaining sides in the proposed triangle may be determined.

Thus to find *ac*, we have the proportion, as the side BC to AC in the computed triangle, so is the side *bc* to *ac* in the triangle proposed, that is, as 90 : 72 : : 73 : 58.4 the side *ac* required; and to find *ab* we have BC : AB : : *bc* : *ab*, or 90: 115 : : 73 : 93.3 the side *ab*. In like manner we may determine, any right angled triangle whatever, provided that one of its acute angles be equal to an acute angle in one of the triangles of the series. It will be easy to see moreover, that we may form a more extensive series of right angled triangles, by making in the first triangle an angle of one degree, in the second an angle of two degrees, in the third of three and so on to eighty nine degrees inclusive; in which case, there will always be found, among the triangles of this series, one similar to any right angled triangle, which may be proposed; provided that one of its acute angles be an entire number of degrees.

In like manner, it is evident, we may so extend the series of computed triangles, that there will always be found, among the number, a triangle similar to the one, which shall be proposed for calculation ; of consequence, as in the preceding example, the parts of this last may be determined from the former by means of a simple proportion.

22. Thus far we have supposed the series of triangles actually constructed, and preserved as in Plate 2, for the calculation of other similar triangles ; such an arrangement, however, would obviously be attended with numerous inconveniences, especially as the series should become more extended. In order therefore to prevent these inconveniences, we may write in a table the different parts in the several triangles of the series, expressing those, which correspond in each by the same general term, and arranging them in a uniform order.

23. To form a table, for example, of the triangles, which we have calculated Plate 2, we may write down in order in the first column, as below, the angles at B in the several triangles, in the second the hypothenuses, in the third the bases, and in the fourth the perpendiculars, recollecting that by the term base we understand the side adjacent the angle at B in each of the triangles, and by the term perpendicular the side opposite this angle.

Deg.	Hyp.	Base	Perp.
10	242	237	43
15	127	123	33
20	121	113	42
25	139	125	59
30	132	114	65
35	125	104	70
40	115	90	72
45	103	73	72
50	93	61	69
55	132	73	109

24. To apply the table; let it be supposed, that we know in the right angled triangle ABC (fig 17) the angle at B 35°, and the side BC 200 feet; in order to solve this triangle, we look along the column of degrees for 35, against this we find 104 the side of the tabular triangle corresponding to the given side in the triangle proposed; then, as 104 the base in the tabular triangle : 125 the hypothenuse of the same triangle : : BC 200 feet : 240.3 feet the hypothenuse of the triangle proposed. And as base 104 : perp. 70: : 200: 134.6 the side AC of the proposed triangle.

SECTION IV.

Of the Method of forming a Series of Triangles, having angles of every possible magnitude.

25. Having seen the use, which may be made of a series of triangles containing angles of every possible magnitude, and the sides of which are calculated; we next inquire into the method of constructing such a series. For the sake of simplicity let the triangles to be determined be right angled. It is evident (fig. 18), that such a series of triangles may be constructed in the quadrant of a circle. For if from each point in the arc AB, we let fall the perpendiculars ED, E′D′ E″D″, &c. upon the radius AC, and draw the radii EC, E′C, E″C, &c., the triangles EDC E′D′C, E″D″C, &c. thus formed are right angled at D, D′, D″ &c., and the angles ECD, E′CD′, E″CD″ &c. will be successively of every possible magnitude. And since any two right angled triangles are similar, when an acute angle in one is equal to one of the acute angles in the other, there can not be proposed a right angled triangle, which will not be similar to some one of the triangles constructed in this manner.

26. To form a table from the triangles of this series, it will be necessary, as before to apply definite terms to ex-

press their sides. The hypothenuses CE, CE′, CE″, &c. being all radii of the same circle, may be designated by the term *radius.* The perpendiculars ED, E′D′, E″D″, &c. manifestly increase with the angles ECD, E′CD′, E″CD″, &c., to which they are opposite, or which is the same thing, with the arcs AE, AE′, AE″ &c., by which these angles are measured. On account of this connexion we designate the perpendiculars ED, E′D′, E″D″, &c., by the term *sine.* We then define the sine of an arc, *the perpendicular let fall from one extremity of this arc upon the radius, which passes through the other extremity.*

In order to fix upon a term to express the remaining side, we now remark, that two *arcs are said to be complements the one of the other, when the sum, or difference of these arcs is equal to the fourth part of the circumference of a circle.* From the definition already given of a sine, the line EF, considered with reference to the arc EB, is the sine of that arc; but EB is the complement of AE; EF therefore is the sine of the complement of AE, and may for the sake of brevity be called its *cosine.* But the side CD of the triangle CDE is equal to EF, since these lines are parallel to each other, and are included between parallels; CD may therefore be considered the cosine of AE, and CD′ of AE′ &c.; whence the remaining side in the triangles CDE, CD′E′, &c. may be designated by the term *cosine*; and we say, that *the cosine of an arc is the sine of the complement of that arc, and is equal to that part of the radius comprehended between the centre and the foot of the sine.*

27. From what has been said, it will now be recollected, that in constructing, in the quadrant of a circle, a series of right angled triangles having angles of every possible magnitude, the radius of the quadrant forms the common hypothenuse in the several triangles, and that the remaining sides

are respectively the sines and cosines of the acute angle, which has its vertex at the centre.

SECTION V.

Of the Method of calculating the sides in the Series of Triangles, constructed in the Quadrant of a Circle.

28. Having ascertained a convenient method of constructing a series of triangles, in which the angles are of every possible magnitude, we proceed to investigate rules for the calculation of the sides. Since in the triangles CDE, CD′E′, CD″E″ it will be sufficient to know the ratio, which the sides have among themselves, it is evident, that we shall not have occasion to calculate the absolute, but merely the relative value of these sides. We may therefore, for the sake of simplicity, consider the radius as unity and proceed to determine the values of the sines and cosines in decimal parts of unity, or which is the same thing, we may consider the radius as divided into 100000 equal parts, and then determine the number of these parts in each of the sines and cosines.

29. It is evident from inspection (fig. 18), that the sine of 90° is equal to radius. And since (fig. 19) the side of an inscribed hexagon is equal to radius (Geom. 271); if through the centre E of the arc BC we draw the radius EF, the chord BC will be bisected at G, and BG the sine of the arc BE will be equal to half of radius; but the arc BE is equal to one third of a quadrant, or thirty degrees; there are then two of the sines in a quadrant, the values of which we may know at the outset, viz. the sine of 90° equal to radius, and the sine of thirty degrees equal to one half of radius.

30. Beginning with the sine BG (fig. 19), we find immediately its cosine GF; for in the triangle BGF, which con-

tains these sides, and the hypothenuse of which is equal to radius or unity, we have $BF^2 = BG^2 + GF^2$; whence $GF = \sqrt{BF^2 - BG^2}$, that is, $\cos EB = \sqrt{R^2 - \sin EB^2}$. To find then the cosine of an arc, when the sine is given; *from the square of the radius subtract the square of the sine, the square root of the remainder will be the cosine.*

31. From what has been said, the relative value of the sides in one of the triangles of the series, which we wish to calculate, may now be considered as known; we are next led to inquire into the use, which may be made of these in the calculation of the remaining triangles of the series.

32. If the arc AE (fig. 20) be divided into two equal parts by the radius CD, the chord AGE will also be divided into two equal parts, and EG will be the sine of the arc DE one half of AE. The triangle AEF, right angled at F, gives $AE = \sqrt{AF^2 + EF^2}$; but $AF = AC - FC = R - \cos AE$, and $EF = \sin AE$; squaring these values of AF and EF and substituting the squares in the expression for AE we have $AE = \sqrt{\sin AE^2 + R^2 - 2R \times \cos AE + \cos AE^2}$; but $EF^2 + FC^2 = CE^2$, or $\sin AE^2 + \cos AE^2 = R^2$, hence $AE = \sqrt{2R^2 - 2R \times \cos AE}$; whence ½ the chord AE, or $EG = \frac{1}{2}\sqrt{2R^2 - 2R \times \cos AE}$. But EG is the sine of one half the arc AE; whence to find the sine of half any arc less than a quadrant; *from twice the square of radius subtract twice the radius multiplied by the cosine of the given arc; one half of the square root of the remainder will be the sine of half this arc.*

33. Let it now be supposed that we wish to calculate a series of triangles, in which there will always be found a triangle having for one of its acute angles any possible number of minutes, or in other words, that we wish to calculate a series of triangles for every minute in the quadrant. By

art. 28 we have radius equal to 1.00000, or, omitting the separatrix, to 100000; the sine of 30° will then be equal (art. 29) to 50000, and its cosine (art. 30) to 86603. By the rule last obtained we may now calculate the sine of 15°, one half of 30°, the sine of 7° 30′, one half of 15°, &c. Proceeding in this manner we obtain at the eleventh division, as below, the sine of $52''\ 44'''\ 3''''\ 45'''''$ equal to 00025, or preserving in the result one more decimal figure, to 000255.

	Sine	*Cos.*
30°	50000	86603
15°	———	———
7° 30′	———	———
3° 45′	06540	99786
———	———	———
———	———	———
———	———	———
$0°\ 14'\ 3''\ 45'''$	00409	99999
———	———	———
———	———	———
———	———	———
$0°\ 0'\ 52''\ 44'''\ 3''''\ 45'''''$	00025	———

We next inquire into the manner, in which the sine of one minute may be obtained from this last.

34. It is evident (fig. 21), that if the arcs *Ab*, *Ac* are very small, they will not differ materially from right lines; the triangles A *b d*, A *c e* may then be considered similar, and *Ab* is to *Ac*, as *bd* to *ce*. *In very small arcs therefore the ratio of the arcs is so nearly equal to the ratio of their sines, that the one may be taken for the other without sensible error.*†

To find the sine of one minute from the sine of $52''\ 44'''\ 3''''\ 45'''''$, we have then the following proportion, $52''\ 44'''\ 3''''\ 45''''' : 60'' :: 000255 : 00029$ the sine of one minute.

35. The sine and cosine of one minute may now be con-

---

† The error in supposing that arcs less than one minute are proportional to their sines, will not affect the first ten places of decimals.

sidered as known, and of consequence all the parts in the first triangle of the series, which we wish to calculate. We proceed to the investigation of principles, by which the remaining triangles of the series may be computed from this first.

Let it be supposed, that the sines of the two arcs AB, AC (fig. 22) are known; it is proposed to find the sines of the sum and difference of these arcs. To construct the figure, make the arc AD equal to the arc AC; draw the chord CD and the radius LA; the radius LA will divide this chord into two equal parts at the point I (Geom. 106); from the points C, A, I and D, let fall upon BL the perpendiculars CK, AG, IH and DF; lastly, at the points I and D, draw IM and DN parallel to BL. Then CK, the sine of BC, the sum of the two arcs AB, AC is equal to KM + MC; but KM is equal to HI; CK is therefore equal to HI + MC, and the sum of the values of HI and MC will be equal to the sine of the sum of the two arcs AB, AC. To find the value of HI, we have LA : LI : AG : HI, or putting for these lines what they severally denote R : cos AC : : sin AB : HI, hence $\mathrm{HI} = \dfrac{\sin \mathrm{AB} \times \cos \mathrm{AC}}{\mathrm{R}}$

And since the triangles LAG, CIM are similar, to find MC we have LA : LG : : CI : MC, that is, R : cos AB : : sin AC : MC, and $\mathrm{MC} = \dfrac{\sin \mathrm{AC} \times \cos \mathrm{AB}}{\mathrm{R}}$

Adding together these two expressions, we have

$$\sin(\mathrm{AB} + \mathrm{AC}) = \frac{\sin \mathrm{AB} \times \cos \mathrm{AC} + \sin \mathrm{AC} \times \cos \mathrm{AB}}{\mathrm{R}}$$

Then since CD is divided into two equal parts at the point I, CN is also divided into two equal parts at the point M, and DF the sine of BD, the difference of the two arcs AB, AC is equal to KN, that is to KM — MC, or which is the same thing to HI — MC. To find then the sine of the difference of the two arcs AB, AC we subtract the one from the other the values of the lines HI, MC already obtained, and

$$\sin(AB - AC) = \frac{\sin AB \times \cos AC - \sin AC \times \cos AB}{R}$$

Having calculated therefore the cosines of any two arcs, the sines of which are given, in order to find the sine of the sum and of the difference of these arcs; *we multiply the sine of the first by the cosine of the second, and the sine of the second by the cosine of the first; the sum of the two products divided by radius will be the sine of the sum of the two arcs; and the difference of the same products divided by radius will be the sine of the difference of these arcs.*

36. The sine of the sum and of the difference of any two arcs being calculated by the rule now obtained, the cosines may be calculated by art. 30. When, however, the sines of any two arcs AB, AC (fig. 22) are given, the cosine of the sum, and of the difference of these arcs may be calculated in a more convenient manner.

Since DC (fig. 22) is divided into two equal parts in I, FK will be divided into two equal parts in H, and LK, the cosine of BC, the sum of the arcs AB, AC will be equal to LH — HK, or to LH — IM. To find LH, we have LA : LI : : LG : LH, that is, R : cos AC : : cos AB : LH, and

$$LH = \frac{\cos AB \times \cos AC}{R}$$

To find IM, the similar triangles LAG, CIM give LA : AG : : CI : IM, that is, R : sin AB : : sin AC : IM, and

$$IM = \frac{\sin AB \times \sin AC}{R}$$

Subtracting this last expression from the former, we have

$$\cos(AB + AC) = \frac{\cos AB \times \cos AC - \sin AB \times \sin AC}{R}$$

Moreover LF, the cosine of BD, the difference of the arcs AB, AC is equal to LH + HF, or since HF is equal to IM, to LH + IM. To find therefore the cosine of the difference of the arcs AB, AC, we add together the values of LH and IM already found, and

$$\cos(AB - AC) = \frac{\cos AB \times \cos AC + \sin AB \times \sin AC}{R}$$

To find therefore the cosine of the sum, and of the difference of any two arcs, whose sines are given, having calculated the cosines of these arcs, *we take the product of the cosines, and also of the sines; the difference of these two products divided by radius will be the cosine of the sum of the two arcs, and the sum of the same products divided by radius will be the cosine of their difference.*

37. From the equation (art. 35) $\sin(AB + AC) = \frac{\sin AB \times \cos AC + \sin AC \times \cos AB}{R}$, we may find the sine of twice any arc, whose sine and cosine are given; for, as the arcs AB, AC may be of any dimensions, they may be considered equal; substituting then AB for its equal AC throughout, we have

$$\sin 2AB = \frac{2 \sin AB \times \cos AB}{R}$$

To find then the sine of twice any arc, whose sine and cosine are given; *we multiply twice the sine of this arc by its cosine, and divide the product by radius.*

38. From the equation (art. 36), $\cos(AB + AC) = \frac{\cos AB \times \cos AC - \sin AB \times \sin AC}{R}$, we may likewise find the cosine of twice any arc, when its sine and cosine are known; for supposing the two arcs equal, as before, and substituting AB for its equal BC throughout, we have

$$\cos 2AB = \frac{\cos AB^2 - \sin AB^2.}{R}$$

Hence, when the sine and cosine of an arc are known, to find the cosine of twice that arc; *we subtract the square of the sine of the arc from the square of its cosine, and divide the remainder by radius.*

39. We have seen (art. 34) the manner in which the sine of one minute may be calculated, consequently, all the parts in the first triangle of the series, which we wish to compute, may be considered as known. By the rules (art. 37, and 38) for the sine and cosine of twice any arc, we may now obtain from this first triangle the sine and cosine of two minutes, or the second triangle, and by the repeated application of the rules (art. 35, and 36) the remaining triangles of the series may be calculated.

Ex. From the data given below it is required to calculate, the sine and cosine of 10°, 11°, &c., extending the calculation to six places of figures, that is, supposing the radius to be divided into 1000000 equal parts.

	Sin.	*Cos.*
1°	017452	999848
5°	087156	996195
10°	——	——
11°	——	——
12°	——	——
13°	224951	974370
14°	——	——
15°	——	——

40. We now perceive the advantage of constructing the series of triangles, which we wish to compute, in the quadrant of a circle; since, if the radius of the quadrant be considered as unity, the value of one of the sides in a triangle of the series will be known, and from this, radius being common to all the triangles, the remaining sides in each of the triangles of the series may be determined.

41. Having calculated a series of triangles by the rules, which we have now obtained, the different parts of this series may be arranged in a table as in art. 23. The radius, however, which is common to all the triangles may be omitted, or written once only at the head of the table.

SECTION VI.

Of the solution of Triangles by the Table of Sines and Cosines.

42. To apply the table of sines and cosines, let there be given for solution the right angled triangle ABC (fig. 23), in which the angle at B is 30°, and the hypothenuse AB three yards.

Since we have considered the radius of the table as unity without reference to any particular denomination of measure, it may manifestly, in all cases, be regarded as a unit of that species, in which the sides of a proposed triangle are expressed. With an extent then equal to one yard, which in the present instance will be the radius of the table, draw from B the arc DF; from D let fall DE perpendicular to BC; then will DE be the sine, and BE the cosine of the angle at B, and DBE will be the triangle of the table, similar to the proposed triangle ABC. To find the side AC, we then have BD : DE : : BA : AC, that is, R : sin B : : BA : AC, and to find BC, BD : BE : : BA : BC, that is, R : cos B : : BA : BC. Since the angle at A is the complement of the angle at B, the cosine of B is the same with the sine of A ; these two proportions may therefore be united into one, and enunciated as follows ; *radius is to the sine of one of the acute angles in a right angled triangle, as the hypothenuse to the side opposite this angle.*

43. Let it be proposed to solve the right angled triangle CDE (fig. 24).

1. Given the hypothenuse CD 250 feet, and the angle at D 30° ; required the remaining parts. To find CE, we have

R : sin D : : CD : CE
or 100000 : 50000 : : 250 ft. : 125 ft.

And to find DE,

R : sin C : : CD : DE
or 100000 : 86603 : : 250 ft. : 216.5075 ft.

2. Given the hypothenuse CD 250 feet, and the side CE 125 feet ; required the remaining parts.

3. Given the side DE 216,5075 feet, and the angle C 30° ; to find the remaining parts.

44. From the preceding example, it is evident, that we may determine, by means of the table of sines and cosines, one of these three things, when two of them are given, viz. *the hypothenuse, a side and one of the acute angles.*

SECTION VII.

Of the second Method of constructing and calculating a Series of Triangles, having angles of every possible magnitude.

45. Let there be given the two sides DE, EC in the triangle CDE (fig. 24), to find the hypothenuse, and one of the acute angles. It will soon be perceived, that the case now proposed cannot be solved by the table of sines and cosines. This table, indeed, may contain a triangle similar to

the triangle proposed; from the data now given, however, no part of that triangle can be known, except the radius or hypothenuse; but neither of the sides DE, EC can be compared with radius; of consequence from the parts now given the remainder cannot be determined.

46. For the solution of this case, it is then evident, that a series of triangles must be so constructed, that the radius shall be a side. In order to construct such a series of triangles, from the extremity of the radius AC (fig. 18) we raise the indefinite tangent AH, and draw from the centre C through each point in the quadrant AB the secants CG, CG′, CG″, &c. It is evident, that the triangles CAG, CAG′, CAG″, &c., right angled at A, must have all the combinations of angles, which can exist in a right angled triangle. The side AC moreover, common to all the triangles, will be the radius of the quadrant.

47. In the application of terms to express the different sides in the triangles of such a series, the common side AC, for the same reason as in the former case, may be called *radius*. The portions AG, AG′, AG″, &c. of the indefinite tangent AH, or in other words the perpendiculars in the triangles CAG, CAG′, CAG″, &c. are the tangents of the arcs AE, AE′, AE″, &c., since we define the tangent of an arc, *that part which is intercepted on the tangent drawn through one extremity of this arc, by the two radii, which terminate it.* The term *tangent* may therefore be applied to designate these perpendiculars. The hypothenuses of the same triangles are secants of the arcs AE, AE′, AE″, &c., since we define the secant of an arc, *the radius, drawn through one extremity of this arc and produced, until it meets the tangent, drawn through the other extremity.* The hypothenuses of the triangles may in consequence be called *secants.*

48. We proceed to develop rules for the calculation of

the sides in this second series of triangles. The triangles CDE, CAG (fig. 18), having the angle C common, are similar; the sides of the latter may therefore be deduced from those of the former.

To find AG, we have CD : DE : : CA : AG, that is, cos AE : sin AE : : R : tang AE, hence

$$\text{tang AE} = \frac{\text{R} \times \text{sin AE}}{\text{cos AE}}$$

In order therefore to find the tangent of an arc, when the sine is given; *we multiply the radius by the sine of the arc, and divide the product by its cosine.*

To find CG, we have CD : CE : : CA : CG, that is, cos AE : R : : R : sec AE, hence

$$\text{sec AE} = \frac{\text{R}^2}{\text{cos AE}}$$

To find therefore the secant of an arc, the cosine being given; *we divide the square of the radius by the cosine of the arc.*

49. From what has been said, it is evident, that having calculated a series of triangles in the quadrant of a circle, in which the radius is the hypothenuse of the triangle, we may by the rules now obtained calculate a second series of triangles similar to the former, in which the radius shall be a side. The parts in this second series may, moreover be arranged in a table, as before, or which is the same thing, the table, already formed from the first series of triangles, may be extended by writing down against the several angles in this table the tangents and secants of these angles.

SECTION VIII.

Of the solution of Triangles by the Table of Tangents and Secants.

50. To show the use, which may be made of the table of tangents and secants, from the point F (fig. 23) draw the perpendicular FG; then will FG be the tangent, and BG the secant of the angle B; BF will be radius, and GBF the tabular triangle similar to ABC.

To find AC, we have BF : FG : : BC : AC, that is, R : tang B : : BC : AC; a proportion, which we may thus enunciate; *radius is to the tangent of one of the acute angles in a right angled triangle, as the side of the right angle adjacent to this acute angle is to the side opposite.*

51. Ex. 1. In the triangle ABC (fig. 25) given the side BC 500 yds, and the angle at B 22° 30′; required the side AC.

To find AC, R : tang B : : BC : AC
or 100000 : 41421 : : 500 : 207.105.

2. Given AC 207.105, and BC 500 yds, to find the angle at B.

By means of the tangents then, we may determine one of these three things, when two of them are known, viz. *the two sides of a right angle and an acute angle.*

52. It remains to show the use, which may be made of the secants. In the triangle ABC (fig. 23) we have BF : BG : : BC : AB, that is, R : sec B : : BC : AB, a proportion, which we thus enunciate; *radius is to the secant of one of the acute angles in a right angled triangle, as the side of the right angle adjacent to this acute angle is to the hypothenuse.*

Ex. 1. In the triangle ABC (fig. 25) given BC 500 yds, and the angle B 22° 30′, to find AB,

R : sec B : : BC : A B

100000 : 1.08239 : : 500 : 541.195.

2. Given AB 541.195 yds, and the angle B 22° 30′, to find BC.

3. Given AB 541.195, and BC 500 yds, to find the angle B.

By means of the secants the same things are determined, as by the sines and cosines ; the secants are in consequence but seldom employed in the solution of triangles.

53. The preceding examples, it will be perceived, do not comprehend the case, in which any two of the sides are given to find the third. This case, however, may be solved by means of the known property of a right angled triangle, viz. the square of the hypothenuse is equal to the sum of the squares of the two sides. It may moreover be resolved with facility by means of the two propositions art. 42, and 50.

Ex. 1. In the triangle CDE (fig. 24) given the side CE 274.703, the side D E 589.101, to find by means of the table the hypothenuse CD. Ans. 650.

Ex. 2. In the triangle CDE (fig. 24) given the hypothenuse CD 745, the side CE 427.317, to find, in like manner, the side DE. Ans. 610.

54. From what has been done, it is manifest, that any right angled triangle whatever may be determined by means of a table constructed in the manner, which has been described, provided that the triangle proposed be similar to some one of the triangles of the table, and that two of its parts be given, exclusive of the right angle.

SECTION IX.

Of Cotangents and Cosecants.

55. For the sake of convenience the table now constructed is usually still further extended, by writing down against the several angles in the table the tangents and secants of their complements.

56. If from the extremity of the radius BC (fig 26), we draw the tangent BF to meet the secant CG, then, since the arc EB is the complement of the arc AE, BF will be the tangent, and CF the secant of the complement of the arc AE; BF may therefore for the sake of brevity, be called the *cotangent*, and CF the *cosecant* of AE.

57. The cotangent and cosecant of the arc AE (fig.26) evidently belong to a triangle distinct from that, in which the tangent and secant are found; the former of these, however, may be deduced from the latter. Indeed, on account of the similar triangles CAG, CBF, we have GA : AC : : CB : BF, that is, tang AE : R: : R : cot AE, hence

$$\text{cot AE} = \frac{R^2}{\text{tang AE}}$$

In order to find the cotangent of an arc, when its tangent is given; *we divide the square of the radius by the tangent of the arc.*

Again, to find FC, we have GA : GC :: BC : FC, that is, tang AE : sec AE : : R : cosec AE, hence.

$$\text{cosec AE} = \frac{\text{sec AE} \times R}{\text{tang AE}}$$

To find therefore the cosecant of an arc, when its tangent and secant are given; *we multiply the secant of the arc by radius, and divide the product by its tangent.*

58. To show the use, which may be made of the cotangents and cosecants, let there be given in the triangle ABC (fig. 27) the side AC 6 yds, the angle at B 40°, to find the remaining side and the hypothenuse. From the point A with an extent equal to one yard describe the arc MK; draw HM perpendicular to AC; AHM will be the tabular triangle similar to the triangle proposed.

1. To find BC we have

AM : HM : : AC : BC
R : Cot B : : AC : BC
100000 : 1.19175 : 6 : 7.1505

2. To find AB, AM : AH : : AC : AB
R : Cosec B : : AC : AB
100000 : 1.55723 : : 6 : 9.3433

3. Given AC 6, and AB 9.3433, to find the angle B.

59. In employing the cotangent and cosecant of B in the preceding example, it is evident, that we have merely used the tangent and secant of A; the work is of consequence the same, as if we had found by subtraction the angle A, and employed the tangent and secant of this angle.

SECTION X.

Miscellaneous remarks on the Trigonometrical Tables.

60. In the preceding articles we have shown the manner, in which a set of trigonometrical tables may be formed. Various methods of abridging the labor of calculation will naturally occur in practice. It is sufficient at present to remark, that from the nature of sines and cosines, tangents, and cotangents &c. if the sines, cosines, tangents &c. have been calculated to 45° inclusive, the remainder of the series may be considered as known.

61 The several triangles of the tables for a given angle

may be exhibited in connection with each other. In the triangle ABC fig. 28), if we consider AC radius, BC will be the cotangent and AB the cosecant of the angle at B; if then with an extent equal to AC we draw the arc DF, and from the points D and F the lines DE, FG perpendicular to BC, then FG will be the tangent and BG the secant of the angle at B; DE moreover will be the sine and BE the cosine of the same angle, and the triangles ABC, GBF, DBE will be the tabular triangles for the angle at B.

62. The radius it will be observed, assumes a different position in each of these triangles; in the triangle DBE, which contains the sine and cosine of the angle at B, it forms the *hypothenuse*; in GBF, which contains the tangent and secant of the angle at B, it becomes the *side adjacent* to this angle, and in ABC which contains the cotangent and cosecant of the same angle it becomes the *side opposite.* This circumstance then must evidently be taken into consideration in calculating a proposed triangle by means of the tables. If the hypothenuse of the proposed triangle be made radius, or if it be compared with the radius of the tables, the *sines* and *cosines* must be employed in the calculation of this triangle; if the side adjacent a given or required angle be made radius, the *tangents* and *secants* must be employed in the calculation; and if the side opposite the given or required angle be made radius, the *cotangents* and *cosecants* must be employed.

63. The numbers in the tables, the construction of which has now been explained, are usually denominated *natural* sines, tangents, &c. By means of these numbers, it has been seen, that the sides and angles of any proposed triangle, which is similar to some one of the triangles of the tables, may be accurately determined; the calculations, however, must be made by the tedious processes of multiplication and

division. To avoid this inconvenience, another set of tables may be constructed by writing down against each of the angles the logarithm of its natural sine, tangent, &c. ; of consequence, in the use of the numbers of such a set of tables, addition and subtraction will take the place of multiplication and division.

64. The logarithmic sines, tangents, &c. are commonly called *artificial* sines, tangents &c. to distinguish them from the natural sines, tangents, &c. In comparing these two sets of tables, one circumstance will require consideration. The radius, to which the natural sines, &c. are calculated, is unity. The secants and a part of the tangents are therefore greater than a unit ; while the sines and another part of the tangents are less than a unit ; of consequence, when the logarithms of these are taken, some of the indices will be *positive*, and others *negative*. To remedy therefore the inconvenience, which would result from the occurrence of both positive and negative indices in the same set of tables, ten is added to each of the indices, by which they are all rendered positive. Thus the natural sine of one minute is 0.00029, or more accurately 0.000290888; the logarithm of this is $\bar{4}.46373$; but the index by the addition of 10 becomes (10—4) or 6 ; of consequence the logarithmic sine of one minute is 6.46373. In like manner the logarithmic sine of two minutes is 6.76476, and so of the rest.

SECTION XI.

Of the solution of oblique angled Triangles.

65. From what has been said, it is evident, that, in order to calculate a triangle by means of the tables, we must be able to construct from the numbers in the tables a triangle similar to the one, which is proposed for calculation.

Let there be given the oblique angled triangle ABC (fig. 29,A), it is proposed to construct from the numbers in the tables a triangle, which shall be similar to this. Having circumscribed a circle about the given triangle, from the centre D of this circle draw the radii DA, DB, DC; with the radius D*b*, equal to that of the tables, describe the circle *abc*; draw the chords *ab*, *bc*, *ac* joining the points of section *a*, *b* and *c*; since D*a* is equal to D*b*, the lines DA, DB are cut proportionally at the points *a* and *b*; *ab* is therefore parallel to AB (Geom. 199); for a similar reason, *ac* is parallel to AC, and *bc* to BC; the triangle *abc* is therefore similar to the triangle ABC (Geom. 209). Upon the side *ab* of the triangle *abc* let fall from D the perpendicular D*e*; since *a*D is equal to the radius of the tables, *ae* will be the sine of the angle *a*D*e*, and *be* the sine of the angle *b*D*e*; produce D*e* to meet the circumference of the circle *abc* in *h*; since the arc *ah* is equal to the arc *hb* (Geom. 105), the angle *a*D*e* is equal to the angle *b*D*e*; the side *ab* of the triangle *abc* is therefore equal to twice the sine of the angle *a*D*e*, or since the angle *a*D*e* is equal to the angle *acb* (Geom. 126), the side *ab* is equal to twice the sine of the angle *acb*; in like manner *bc* is equal to twice the sine of the angle at *a*, and *ac* to twice the sine of the angle at *b*; but the angles *a*, *b* and *c* in the triangle *abc* are equal respectively to the angles A, B and C in the triangle ABC; the sides *ab*, *bc*, *ac* in the triangle *abc* are therefore equal respectively to twice the sine of the angles C, A and B opposite the homologous sides AB, BC, AC in the triangle ABC.

With respect to the side *ab* in the case, where the angle C is greater than a right angle (fig. 29, B); since from the definition of a sine, an angle will have the same sine with its supplement to two right angles, *ae*, the sine of the angle *a*D*e*, will also be the sine of the angle *a*D*k*;

and the side *ab* will be equal to twice the sine of the angle *aDk*; but the angle *a*D*k*, or, which is the same thing, the angle ADE is equal to the angle ACB; the side *ab* is therefore equal, as before, to twice the sine of the angle at C. From the numbers in the tables we may then always construct a triangle similar to any oblique angled triangle, which may be proposed for calculation, by taking for each of the sides in the required triangle twice the sine of the angle opposite the homologous side in the triangle proposed. The tables, therefore, which at first appear limited to the solution of right angled triangles, may evidently be applied to the solution of triangles of whatever kind.

66. We proceed to the investigation of rules for the calculation of plane triangles of whatever kind by means of the tables. The similar triangles ABC, *abc* (fig. 29) give AB : BC : : *ab* : *bc* &c., that is, AB : BC : : 2 sin C : 2 sin A, or, sin C : sin A ; a proportion, which, it is easy to see, is of general application, and which may be enunciated, as follows ; *in any triangle whatever, the sines of the angles are to each other as the sides opposite to these angles.*

67. Case 1. To apply this principle, let there be given in the triangle ABC (fig. 30) the side BC 70 yds, the angle A 86°, and the angle C 45°, to find the remaining parts.

1. To find AB,	sin A 86°	9.99894
	BC 70 yds	1.84510
	sin C 45°	9.84949
		11.69459
	AB 49.62 yds	1.69565

2. In like manner, AC is found equal to 52.96 yds.

Case 2. Given two of the sides, viz. AB 49.62 yds, BC 70 yds, and the angle A opposite BC 86°, to find the remaining parts.

1. To find the angle at C.

BC 70 yds	1.84510
sin A 86°	9.99894
AB 49.62 yds	1.69565
	11.69459
sin C 45°	9.84949

2. Having obtained the angle C, the side AC is found as in the preceding case.

3. Given in the triangle ABC (fig. 13) the side AB 72 feet, the side AC 41.5 feet, and the angle B 30°, to find the angle C.

AC 41.5	1.61805
sin B 30°	9.69897
AB 72	1.85733
	11.55630
sin C 60° 10′	9.93825

In this last example the solution (art. 16) is ambiguous; since, however, the angles ACB, AC′B are supplements, the one of the other, and the sine is the same for an angle, and its supplement; if the angle 60° 10′ first obtained does not answer the conditions of the question, we then take its supplement 119° 50′ for the angle C, which is sought.

68. From the preceding examples, it appears, that we may always solve a triangle by the rule just obtained, 1. ***when a side and two of the angles are given; 2. when two of the sides and an angle opposite to one of them are given.***

69. In the triangle ABC (fig. 30) let there now be given the sides AC and BC, and the angle C contained between them, to find the remaining parts. This case, it will be perceived, does not admit of solution by the preceding rule; a new one must therefore be sought. Having described a circle with a radius equal to that of the tables (fig. 31), draw the diameter AM; let the arc AB, set off from A, be equal to the number of degrees, contained in the angle at A in the

proposed triangle ; and also the arc AC to the number of degrees in the angle at B ; from B draw the chord BD perpendicular to AM ; from C draw CP perpendicular and CF parallel to AM ; join BF, FD ; from F, with the radius of the circle ABD equal to that of the tables, draw the arc IGK meeting CF in G, and, at the point G, draw HL perpendicular to CF ; the line GL is the tangent of the angle GFL, or which is the same thing, of CFD, and GH is the tangent of GFH, or CFB ; and the angles CFD, CFB having their summets in the circumference are measured by half the arcs CD, CB, on which they stand ; but CD is the sum of the two arcs AB, AC, and CB is the difference of these two arcs ; whence GL is the tangent of the half sum, and GH the tangent of the half difference of the arcs AB, AC ; but the arcs AB, AC, by hypothesis, are respectively the measures of the angles A and B in the proposed triangle ABC (fig. 30) ; GL is therefore the tangent of the half sum and GH the tangent of the half difference of the angles A and B; in like manner it may be shown, that DE is the sum, and BE the difference of the sines of the same angles A and B ; wherefore on account of the parallel lines BD, HL we have DE : BE : : LG : GH, or putting for these lines what they severally denote

$$\sin A + \sin B : \sin A - \sin B :: \text{tang}\,\frac{A+B}{2} : \text{tang}\,\frac{A-B}{2};$$

but by art. 66 we have sin A : BC : : sin B : AC, hence

$$\sin A + \sin B : \sin A - \sin B :: BC + AC : BC - AC :$$

$$\text{whence } BC + AC : BC - AC :: \text{tang}\,\frac{A+B}{2} : \text{tang}\,\frac{A-B}{2};$$

a proportion, which we may thus enunciate ; *the sum of two sides of a triangle is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

70. Ex. 1. Let AC (fig. 30) be 52.96 yds, BC 70 yds, and the angle C 45°; it is required to find the remaining parts. Subtracting the angle C 45° from 180°, and dividing the remainder by 2, we have

$\frac{A+B}{2} = 67° \, 30'$, and to find $\frac{A-B}{2}$ we have

$BC + AC$	122.96	2.08976
$BC - AC$	17.04	1.23147
tang $\frac{A+B}{2}$	67° 30′	10.38278
		11.61425
tang $\frac{A-B}{2}$	18° 30′	9.52449

Having obtained the half difference between A and B, the greater angle A 86° is found by adding the half difference to the half sum, and the less angle B 49° by subtracting the half difference from the half sum. The angles being thus obtained, the remaining side may be calculated as before.

Ex. 2. Given two sides of a plane triangle 450 and 540, and the included angle 80°, to find the remaining parts.

Ans. Angles 56° 11′, 43° 49′, and side 640.08

Ex. 3. Given two sides of a plane triangle 76 and 109, and the included angle 101° 30′, to find the remaining parts.

Ans. Angles 30° 57′ 30″, 47° 32′ 30″, and side 144.8.

71. The preceding rules do not include the case, in which the three sides are given to find the angles. To obtain an expression for this case, in the triangle ABC (fig. 32) let fall from the point B, BD perpendicular to AC, and from the same point, with a radius equal to the side BC, describe the circumference CEHF; extend the side AB, until it meets the circumference in E; then, since AE and AC are drawn from the same point A without the circle, AC : AE :: AG : AF (Geom. 225); but AE is equal to AB + BC, and AG is equal to AB — BC; AF moreover is equal to AD — DC;

therefore $AC : AB + BC :: AB - BC : AD - DC$. If then upon the longest side of a triangle we let fall a perpendicular from the opposite angle, we have the following proportion; *as the longest side of a triangle is to the sum of the two other sides, so is the difference of these last to the difference of the segments made by the perpendicular.*

72. Ex. 1. Given the three sides of a triangle ABC (fig.33), viz. AB 66 feet, AC 75, and BC 34 feet, to find the angles. By the preceding rule

AC 75	1.87506
AB + BC 100	2.00000
AB — BC 32	1.50515
	3.50515
AD — DC diff. seg. 42.67	1.63009

Adding half the difference of the segments to half the sum, we obtain the greater segment AD 58.83 feet; subtracting the half difference from the half sum, we obtain the less segment DC 16.17 feet.

Then to find the angle A, $AB : R :: AD : \cos A$ 26° 57′; and to find the angle C, $BC : R :: DC : \cos C$ 61° 36′.

Ex. 2. The sides of a plane triangle are 40, 34 and 25 feet respectively; required the angles.

Ans. 38° 25′ 20″, 57° 41′ 25″, 83° 53′ 15″.

Ex. 3. The sides of a plane triangle are 390, 350 and 270 feet respectively; required the angles.

Ans. 42° 21′ 57″, 60° 52′ 42″, and 76° 45′ 21″

SECTION XII.

Mensuration of Heights and Distances.

72. The preceding principles may be readily applied to the mensuration of the heights and distances of objects, in cases where a direct measurement cannot be made or where it cannot be made with convenience. The following are among the problems which most frequently occur in practice.

PROBLEM I.

73. To determine the altitude BC of a steeple or other object (fig. 37), situated on a horizontal plane.

From the bottom of the steeple measure, in a direct line upon the horizontal plane, any convenient distance BA ; at the extremity A of this line measure the angle BAC, comprised between the line BA and an imaginary line, drawn from the point A to C the top of the steeple ; then in the triangle ABC the side AB and all the angles will be known; whence to find BC, we have

$$\sin ACB : AB :: \sin BAC : BC$$

$$\text{or} \quad R : \text{tang } BAC :: AB : BC$$

The angle BAC situated in a plane perpendicular to the horizon, and comprised between a horizontal line AB and an ascending line AC, is called an angle of *elevation*.

The line AB, which has been measured in order to determine by means of it the required height, is called a *base line*.

Ex. 1. Let the base line AB be 200 feet, and the angle of elevation BAC be 47° 30′ ; required the height of the steeple. Ans. 218.26 feet.

Ex. 2. Let AB be 384 feet, and the angle at A 36° 52′; required the height of the steeple. Ans. 288 feet.

PROBLEM II.

74. To determine the perpendicular height of a cloud or other object above a horizontal plane.

At two convenient stations A and B on the same side of the object (fig. 38) or on opposite sides (fig. 39) and in the same vertical plane, let two observers take at the same time the angles of elevation CAB, CBD, and let the distance BA be measured, then the exterior angle of a triangle being equal to the two interior and opposite, subtracting (fig. 38), the angle CAB from CBD, we have the angle ACB;

whence sin ACB : AB : : sin CAB : BC

then in the triangle CBD, we have to find CD the height required R : BC : : sin CBD : CD

or (fig. 39) subtracting the sum of the angles at A and B from two right angles, we have the angle ACB; to find one of the remaining sides in the triangle ACB, AC for example, we have sin ACB : AB : : sin ABC : AC

then in the triangle ADC, to find CD we have

R : AC : : sin CAD : CD

Ex. 1. Let the angles of elevation (fig. 38) be 31° and 46° respectively, and the line AB 100 yards; what is the height of the cloud?

We have in this example ACB = 15°, then

	sin ACB 15°	9.41300
	AB 100	2.00000
	sin CAB 31°	9.71184
	BC	2.29884
Then	Radius	10.00000
	BC	2.29884
	sin CBD 46°	9.85693
	CD 143.14	2.15577

Ex. 2. Let the angles of elevation (fig. 39) be 53° and 79° 12′ respectively, and the distance AB 100 rods; what is the altitude of the cloud? Ans. 105.89 rods.

PROBLEM III.

75. To find the height of an inaccessible object standing on a horizontal plane.

At two stations A and B in the same vertical plane passing through the top of the object (fig. 40), take the angles of elevation CAB, CBD, and measure the distance AB, then in the triangle CAB we have

$$\sin ACB : AB :: \sin CAB : CB$$

whence to find CD the altitude required, we have in the triangle CBD $R : CB :: \sin CBD : CD$

Ex. 1. Let the angles of elevation be 32° and 58°, and the base line 100 yards; required the height of the object.

Ans. 102.51 yards.

Ex. 2. Let the angles of elevation be 40° and 60°, and the base line 100 feet; required the height of the object.

Ans. 162.75 feet.

PROBLEM IV.

76. To find the height of an accessible object standing on an inclined plane.

From the bottom of the object (fig. 41) measure in the same direct line any two distances AB, BD, at B take the angle CBA, and at D the angle CDA, then in the triangle CDB we have $\sin DCB : DB :: \sin CDB : BC$

BC being thus determined, we have in the triangle CBA

$$BC + BA : BC - BA :: \text{tang } \tfrac{1}{2}(A + C) : \text{tang } \tfrac{1}{2}(A - C)$$

having found by means of this proportion one of the remaining angles in the triangle CBA, BCA for example, we have, to find the height required

$$\sin BCA : BA :: \sin ABC : AC$$

Ex. Let the distances AB, BD be 40 and 60 feet, the angle at B 41°, that at D 23° 45′ ; required the height of the object. Ans. 57.624 feet.

PROBLEM V.

77. To find the distance between two objects inaccessible by reason of an intervening river, the objects being both on the same level.

At two stations A and B (fig. 42) also on a level, we take the angles CAD, DAB, CBA, DBC and measure the distance AB ; then in the triangle ACB we have

sin ACB : AB : : sin CAB : CB

and in the triangle ADB

sin ADB : AB : : sin DAB : DB

then in the triangle CBD we have

$$BC + BD : BC - BD : : \text{tang} \tfrac{1}{2} (D + C) : \text{tang} \tfrac{1}{2}(D - C)$$

having found by means of this proportion the angle BCD for example, we have finally for the distance sought

sin BCD : BD : : sin CBD : CD

Ex. Let the angles in the order stated above be 37°, 58° 20′, 53° 30′, 45° 15′, and the base line 300 yards. What is the distance between the two objects ? Ans. 479.79 yards.

PROBLEM VI.

78. The height of a tower being given to find the horizontal distance between two objects, situated on the same level with the tower and in a direct line from the bottom of it.

Take the angles EAD, EAC (fig. 43) contained between the line EA parallel to the horizon and lines drawn from A, the top of the tower, to the objects D and C respectively, then in the triangle DAB all the angles and the side AB will be known, from which DB may be found; in like manner in the triangle ACB all the angles and the side AB will be known from which CB may be found; the value of these being determined, the latter subtracted from the former will give the distance sought.

An angle EAD comprised between a line parallel to the horizon and a descending line is called an angle of *depression.*

Ex. Let the height of the tower be 120 feet, the angle of depression of the nearest object 57°, that of the most remote 25° 30′. What is the distance between the two objects;

Ans. 173.65 feet.

PROBLEM VII.

79. To find the altitude of a hill above a horizontal plane passing through a given point, on the side of another hill opposite.

At the given point B (fig. 45) take the angle of elevation CBD of the top of the hill, the altitude of which is sought, from B measure a base line BA directly up the other hill, and at A take the angle of elevation CAE and the angle of depression EAB, then in the triangle CAB all the angles will be given and the side AB to find CB; and CB being found, we shall have in the triangle CBD all the angles and the side CB to find CD the altitude sought.

Ex. Let the angle CBD be 5° 52′, the base line AB 642 yards, and the angles CAE, EAB 3° 59′, and 39′ respectively. What is the altitude of the hill? Ans. 161.3 yards.

PROBLEM VIII.

80. To find the distance between two inaccessible objects, both of which can be seen at the same time from one point only.

Let D (fig. 46) be the point from which both objects can be seen at the same time; in any convenient directions take a station, C where A can be seen, and a station E, where B can be seen, and measure the distances CD, DE; at D take the angles ADC, ADB, BDE, and at C and E the angles ACD, BED; then in the triangles ACD, BDE we shall be able to determine the sides AD, DB, and these being found, we shall have in the triangle ADB data sufficient to determine AB the distance sought.

Ex. Let CD be 200 yards, the angle ADC 89°, ACD 50° 30′; let DE also be 200 yards, the angle BDE 54° 30′, BED 88° 30′, and ADB 72° 30′; required the distance between the two objects. Ans. 345.5 yards.

PROBLEM IX.

81. To find the altitude of an object, situated on the summit of a hill, above a horizontal plane at the foot of the hill.

Having assumed a station B (fig. 47) upon the horizontal plane, measure in a direct line from the object a base line BA, at B take the angles of elevation DBE, CBE, and at A the angle of elevation CAE; in the triangle CAB we determine the side CB, then in the triangle CBE we find the remaining sides CE and BE; this being done, in the triangle DBE we find the side DE; subtracting next DE from CE already found we have the altitude CD sought. CD may also, it is evident, be found by means of the triangle CBD without calculating CE and DE.

Ex. Let the angle DBE be 40°, CBE, 51°, the base line BA 100 yards, and the angle CAE, 33° 45′, to find the altitude of the object. Ans. 46.67 yards.

PROBLEM X.

82. To determine the distance of an inaccessible object, when an instrument for measuring angles cannot be procured.

Let E be the object (fig. 50) and AE the distance sought, from the station A in the direction EA, measure any convenient distance AC; assume another station B, and in the direction EB measure any convenient distance BD; measure also the distances AB, AD and BC. In the triangles ACB, ABD we shall have the sides given to determine the angles, and these being found, those of the triangle AEB will be known; then in the triangle AEB we have the angles and a side to find AE the distance sought.

Ex. Let AB be 500, AC 100, CB 560, BD 100, and AD 550 yards; required the distance AE.

Ans. 536.25 yards.

QUESTIONS AND PROBLEMS FOR PRACTICE.

1. A line 27 yards long will exactly reach from the top of a fort to the opposite bank of a river, known to be 23 yards broad; what is the height of the wall? Ans. 42.42 feet

2. The height of an object on a horizontal plane is 200 feet, and its angle of elevation at the place of the observer is 42° 30′; what is his distance from the object?

Ans. 218.26 feet.

3. A ladder 193.55 feet long is placed against a wall in an oblique position, and reaches to a point 75.83 feet from the ground; what is the angle at which it is inclined to the ground? Ans. 23° 3′ 55″.

4. From the edge of a ditch 18 feet wide, surrounding a fort, I took the angle of elevation of the top of the wall and found it 62° 40′; required the height of the wall, and the length of a ladder necessary to reach from my station to the top of it. Ans. Height 34.82, length 39.2 feet.

5. A ladder 125 feet long is placed against a wall so that the angle at the bottom is double the angle at the top; how high up the wall does it reach and how far distant from the wall is its foot? Ans. 108.25, and 62.5 feet respectively.

6. A person attempts to swim across a river, whose breadth is 329 yards. How far from the place immediately

opposite to his departure does he arrive, if the stream forces him to swim twice as far as he would have done, had there been no current? Ans. 569.9 yards.

7. From the top of a tower, 143 feet high, by the sea side, I observed that the angle of depression of a ship's bottom, then at anchor, was 55°; what was its distance from the bottom of the wall? Ans. 100.13 feet.

8. Wanting to know the breadth of a river, I measured 100 yards in a right line close by one side of it, and at each end of this line I found the angles subtended by the other end and a tree close by the other side of the river to be 53° and 79° 12′. What is the perpendicular breadth?

Ans. 105.89 yards.

9. Being on one side of a river and wanting to know the distance to a house, which stood on the other side, I measured 200 yards in a right line by the side of the river, and found that the two angles at each end of this line formed by the other end and the house were 73° 15′ and 68° 2′; what was the distance between each station and the house?

Ans. 296.54 and 306.19 feet.

10. A gentleman wishing to ascertain the distance between two trees A and B, which could not be directly measured on account of a pool, that occupied the intermediate space, assumed a station C, from which both could be seen, and found by measurement the distance AC 7.35 chains, the distance BC 8.4 chains, and the angle ACB 55° 40′. What is the distance of the trees A and B. Ans. 7.412 chains.

11. From the top of a tower, whose height is 108 feet, the angles of depression of the top and bottom of a vertical column standing in the horizontal plane are found to be 30° and 60° respectively; required the height of the column.

Ans. 72 feet.

12. From a window B near the bottom of a house sup-

posed to be on a level with a monument CD, the angle of elevation CBD of the top of the monument being 40°, and from another window A 18 feet higher the angle of elevation CAE being 37° 30′, it is required to find the height of the monument CD and its distance BD.

Ans. CD 210.44 feet, and BD 250.79 feet.

13. Wanting to know the height and distance of an object on the other side of a river, and on a level with the place where I stood, close by the side of the river; not having room to go backward on the same plane, on account of the immediate rise of the bank, I placed a mark where I stood and measured in a direct line from the object up the hill a distance of 132 yards; I then found the angle of depression of the mark by the river's side 42°, that of the bottom of the object 27°, and of its top 19°; required the height of the object, and the distance of the mark from its bottom.

Ans. Height 28.63, distance 75.25 yards.

14. It is required to find the height of a castle AB, situated on an eminence by the sea shore, above the level of the sea and its horizontal distance from a ship S at anchor, the angles of depression, EAS, FBS being given equal respectively to 4° 52′, and 4° 2′, and the height of the castle being 54 feet.

Ans. Distance 3690 feet, and height 314 feet.

15. At a station A at the bottom of a hill, I took the angle of elevation of the top of an object D on the summit of the hill, 38°, and measuring directly up the hill a distance of 676.47 feet to another station B, I took at this station the angle of elevation of the top of the object D, 46°, and the angle of depression of the first station A, equal to the angle of elevation of the second station from the first, 27° 30′ 25″; what is the height of the object D above the level of the first station A. Ans. 949.19 feet.

16. Suppose the object CD (fig. 45) to stand upon a horizontal plane ABC and that AB is equal to 250 yards, and that the angles at its extremities are known, viz. CAB 56° 46′, CBA 62° 54′, DAC 6° 40′, DBC 7° 6′, What is the height CD and the two distances AD and BD ?

Ans. AD 254.989, BD 238.814, and CD 29.745 yards.

17. To ascertain the distance between two places, the angle which they subtended at a point equally distant from both was observed, and again at another point equally distant from both ; and the distance between the points of observation was measured. To determine from these data the distance of the places, 1° when the points of observation are both on the same side of the places, 2°, when they are on opposite sides.

18. At a given distance from an obelisk, whose height is known, a colossal statue on the top of it subtends the same angle as the observer, when seen from the base of the obelisk. Supposing the obelisk and the observer on the same horizontal plane, and the height of the observer known, show how the height of the statue may be determined.

19. Two stations on the side of a mountain are within sight of each other, of a signal D placed upon the summit of the mountain and of another E upon the horizontal plane ; supposing the distance between the two stations given, explain the method of finding the height of the mountain by means of observations made at these two stations, 1° when the stations are in the same vertical plane with the two signals, 2° when they are in different planes.

20. Two spectators at two different stations on a horizontal plane, the distance between which is known, observe at the same instant a balloon, which rises into the air at a uniform rate, and after a given interval of time again observe

it; what are the observations necessary to determine the two elevations of the balloon and its rate of moving, and how shall the data be applied to this purpose, 1°, when it ascends in a perpendicular line and the stations are not in the same vertical plane with it, 2°, when it ascends in an oblique line and the stations are in the same vertical with it.

SECTION XIII.

Of the Line of natural Sines, Tangents, &c.

83. Let the quadrant AB (fig. 35) be divided into equal portions, of a degree each, for example; and let lines be drawn, parallel to AC, from each point of division in the quadrant AB to the line CB. The several distances from C to the points, where these parallels meet the line CB, will be equal respectively to the lengths of the sines for each degree in the quadrant AB. Thus the extent from C to 10 will be equal to the sine of 10°, from C to twenty to the sine of 20°, and so on. The line CB, divided in this manner, is called *a line of sines.*

In like manner if through each of the points of division in the quadrant AB secants be drawn to meet the indefinite tangent BD, the several distances from B to the points of division in BD will be equal respectively to the tangents for each degree in the quadrant AB, and the line BD will be a *line of tangents.*

If moreover the different distances from the centre C to each of the divisions on the line of tangents BD be transferred to the line CE, the line CE, will be a *line of secants.* Thus B 20 is the tangent, and C 20 the secant of 20°.

84. A single example will show the use, which may be made of these lines.

In the triangle ABC (fig. 30), given the side BC 70 yds, the angle A 86°, and the angle C 45°, to find AB.

By art. 66, sin A 86° : sin C 45° : : BC : AB. In order therefore to determine the value of AB by the line of sines, we measure by means of a convenient scale of equal parts, the extent from *C* to 86 on the line of sines (fig. 35), and also the extent from *C* to 45 ; we then have the relative values of the first and second terms in the above proportion.

Let the sine of A determined in this manner be 332, and the sine of C 235, then

332 : 235 : : 70 : 49.5 the side AB.

The method of solving triangles, which has now been exhibited, is obviously the same, as that by the table of natural sines, &c, with the exception that the values of the sines in the present case are determined by geometrical construction.

SECTION XIV.

Of the Line of Logarithmic Sines, Tangents, &c.

85. The logarithms of numbers may be conveniently represented by lines. Thus, since the logarithm of 10 is 1 and the logarithm of 100 is 2, if a line of one foot be made to represent the logarithm of 10, a line of two feet will represent the logarithm of 100.

For the purpose of forming a logarithmic scale let the line *ab* (fig. 36), intended for the scale and taken of any length at pleasure, represent the logarithm of 100 ; then 100 will stand at the end of the scale ; and 1, the logarithm of which is 0, will stand at the beginning of the scale. For the intermediate numbers 1, 2, 3, 4, &c, it is easy to see, that we have merely to set off in order from the beginning of the scale the several lengths, which shall be equal respectively to the logarithms of these numbers, when *ab* is equal to the logarithm of 100. The logarithm of 100, taken from a common table of logarithms but extending to three places only of decimals, is 2.000, or, omitting the separatrix, 2000. Under the same circumstances the logarithm of 2 is 301, and the logarithm of 3 is 477, &c. Let it be supposed, that the line *ab* is divided into 2000 equal parts; then, as *ab* has been made to represent the logarithm of 100, 301 of these parts will represent the logarithm of 2. To determine therefore the place of the number 2 on the scale, we set 301 of the parts, into which *ab* is supposed to be divided, from 1 to 2; to determine the place of 3 we set off in like manner 477 of the same parts, and so of the rest. Proceeding in this manner we determine the primary divisions of the scale, and to obtain the intermediate divisions we set off in a similar manner the logarithms of the intermediate numbers. Thus, when the logarithm of 100 is 2000, the logarithm of 1.1 is 41, the logarithm of 1.2 is 79, &c. These numbers being set off in order from the beginning of the scale will divide the first primary division into 10 parts. In like manner the remaining primary divisions may be further divided.

86. The logarithmic scale, the construction of which has now been explained, contains the logarithms of numbers from one to a hundred only. The decimal part of the logarithm of any number, it will be recollected, is the same as that of the number, when multiplied or divided by 10 or 100, &c. We may then consider the point under the 1 at the beginning of the scale to represent the logarithm of 1, or 10, or 100, or $\frac{1}{10}$, or $\frac{1}{100}$, &c. Hence the same scale may easily be made to answer for all numbers whatever.

87. From what has been said the logarithm of any given number may with facility be taken from the scale.

Let it be required to take from the scale the logarithms of the following numbers, viz. 75, 43.5, 365, 2450, 10844.

88. Multiplication, division, &c. are performed by the line of numbers on the same principles as by common logarithms.

Multiplication.

1. Let it be required to multiply 6 by 8. Since addition in logarithms takes the place of multiplication, the extent from 1 to 6 added to the extent from 1 to 8 will be equal to the extent from 1 to 48 the product.

To multiply then by the logarithmic line, we take off with the compasses that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line, which represents the logarithm of the product.

2. It is required to multiply 9 by 7, 18 by 5, 44 by 63, 120 by 75.

Division.

It is required to divide 56 by 7, 120 by 12, 400 by 50, 2880 by 320.

Proportions.

1. To find the fourth term of the proportion, in which 2, 4 and 8 are the first three terms.

2. To find the fourth term of the proportion, in which 15, 75 and 40 are the first three terms.

3. Let 4324, 540.5 and 2560 be the three first terms of a proportion; required the fourth term.

89. We proceed to construct a line of logarithmic sines, that shall correspond with the line of numbers. For this purpose let the line *cd*, intended for the scale, be taken (fig. 36) equal to *ab*. In order that this scale may corres-

pond with the line of numbers, it will be obvious, that the difference between the extreme indices on both should be the same. On the line of numbers the difference between the extreme indices is 2. The logarithmic sine of 0° 34′ 22″ 41‴, is 8.000, and that of 90° is 10.000 the difference of the indices being 2. If then the point *c* at the beginning of the scale be marked for the place of 0° 34′ 22″ 41‴ the point *d* at the end of the scale will be the place for 90°. In order to find the intervening divisions we suppose the line *cd*, in the same manner as has been supposed with respect to *ab*, to be divided into 2000 equal parts. Then, since the difference between the logarithmic sine of 0° 34′ 22″ 41‴ and the logarithmic sine of 1° is 241, the extent from *c* to 1 equal to 241 of the parts, into which *cd* is supposed to be divided, will give the point of division for the sine of 1° ; in like manner the extent from 1 to 2, equal to 542 of the same parts, will give the point of division for 2°, and so of the rest. The scale may also be constructed by marking the point *d* for the sine of 90°, and setting the arithmetical complement of each degree &c. backward from *d* toward *c*.

90. The logarithmic tangent of 0° 34′ 22″ 35‴ is 8.000, and that of 45° is 10.000. To form therefore a line of logarithmic tangents corresponding to the line of numbers, let the point *e*, on the line *ef* taken equal to *ab*, be marked for the tangent of 0° 34′ 22″ 35‴, and the point *f* for the tangent of 45° ; then, *ef* being supposed also to be divided into 2000 equal parts, the intervening points of division may be marked as before.

For the points of division above 45° the scale should extend much further to the right; but (art. 57) $\frac{\mathrm{R}}{\mathrm{tang}}=\frac{\mathrm{cot}}{\mathrm{R}}$, that is, in logarithms $\mathrm{R}-\mathrm{tang}=\mathrm{cot}-\mathrm{R}$. The logarithmic tangent therefore of an arc below 45° is as much less than that of 45°, as the logarithmic cotangent of this arc is greater than that of 45°. Instead then of extending the scale further to the right, the numbering after reaching 45° may be continued back from right to left, and the same point of division be made to answer for an arc and for its complement.

91. We shall now show the use, which may be made of the logarithmic lines in the solution of triangles.

Ex. 1. In the triangle ABC (fig, 30), given the side AB 40 yds, the side AC 90 yds, and the angle C 20°, to find by means of the logarithmic lines the angle B.

By art. 66, AB : AC : : sin C : sin B ; wherefore in logarithms AC — AB = sin B — sin C ; hence to find the angle at B, we take in the compasses from the line of numbers the extent from 40 to 90 ; this extent will reach from 20° on the line of sines to 50° 20′ the answer.

2. Let the angle C equal 55°, the angle B 32°, and the side AB 100 feet to find AC.

3. Given AB 53, BC 13, and the angle B, 120°, to find AC.

92. The scale fig. 36, the construction of which has now been explained, is commonly called *Gunter's scale* from the name of its inventor. It furnishes a ready method of solving triangles in those cases, where great accuracy is not required. The mode of solution differs from that by the common tables of logarithms in this respect only, that the logaithmic values of the sides and angles are expressed by lines instead of numbers.

SECTION XV.

Of Trigonometrical Analysis.

93. In the preceding sections we have treated of sines, cosines, tangents, &c. merely in their relation to the calcula-

tion of the sides and angles of triangles. The use of these lines may however be extended to other objects. In conducing the investigations of physical astronomy in particular important assistance is derived from them; because as it is only the angular positions of the heavenly bodies, which we observe, all disquisitions concerning the motions, orbits, &c. of these bodies must necessarily involve the sines, cosines, tangents, &c. of angles.

94. In the solution of plane triangles the arcs, which fall under consideration, never exceed a semicircumference; and we have occasion to consider merely the numerical values of the sines, cosines, tangents, &c. In the more extended applications of these lines, however, the arcs, which occur, assume every possible magnitude, and the sines, cosines, &c. being regarded as *functions* of their corresponding arcs, that is, as analytic expressions, the values of which depending entirely upon those of the arcs may be deduced from them by certain arithmetical operations, we shall have occasion to consider not only the changes that take place in the numerical values of the different trigonometrical lines, but also the changes that occur in their positions as the arcs to which they belong vary in magnitude.

95. To understand clearly the nature of these changes let us attend to the variations, which take place in the numerical values, and in the positions of the sines, cosines, tangents, &c. of an arc, while it increases from 0 to an entire circumference.

In the circle ABDE (fig. 34) let the diameters AD, BE be drawn perpendicular to each other; the circle will then be divided into quadrants. To trace the changes in question, let the radius CF, at first coinciding with CA, be supposed to revolve about the point C, as upon a pivot, and departing from the position CA, so that the arc AF shall have successively all magnitudes whatever fom 0 to an entire circumference, and the angle ACF shall increase from 0 to four right angles. In the first quadrant, when the radius CF coincides with CA, so that the arc AF is 0, the sine GF, it is evident, is also 0, while the cosine CG is equal to radius or unity. When the radius CF moves off from CA, the sine GF increases as the point F advances toward B, until, when the point F has arrived at B, the sine GF coincides with CB, and becomes equal to radius or 1. At the point B the arc AF is equal to a quadrant, and the angle ACF to a right angle. Under the same circumstances the cosine CG constantly decreases, until when the point F coincides with B, the cosine becomes equal to 0. In the second quadrant as the point F′ moves on from B to D, the sine G′F′ decreases, and the cosine CG′ increases, until, when the point F′ coincides with D, and the arc AF′ becomes equal to a semicircumference, the sine G′ F′ is 0, and the cosine C G′ is equal to radius or 1.

It may be remarked in passing, that the lines G′F′, CG′ are respectively the sine and cosine of the arc DF′, and also of its supplement AF′; whence the absolute magnitude of the sine and cosine of an obtuse angle is the same with that of its supplement.

In the third quadrant, as the point F″ moves on from D to E, the sine G″ F″ increases, and the cosine CG″ decreases, until at the point E, where the arc AF″ becomes equal to three fourths of a circumference the sine is equal to 1, and the cosine to 0. From E to A the sine G‴F‴ decreases and the cosine CG‴ increases, until at A, when the whole revolution has been completed, and the arc AF‴ becomes equal to an entire circumference, the sine is reduced to 0, and the cosine becomes equal to 1.

In the first and second quadrants the sines, being all situated on the upper side of the diameter AD, from which they are measured, are considered *positive*; in the third and fourth quadrants falling below the diameter AD, and assuming in consequence a direction contrary to that, which they had before, they are considered *negative*. In the first and fourth quadrants the cosines, which are all situated on the left of the point C, to which they are referred, are considered *positive*; in the second and third quadrants falling on the right of C, and assuming a direction contrary to their former direction, they are considered *negative*

96. Observing in like manner the changes, which occur in the magnitude of the tangents through the course, which has been described, we find, that from A to B the tangents increase continually, as the arc AF increases; at the point B where the arc AF becomes equal to a quadrant, the secant CM coinciding with CB is parallel to the tangent AM, and therefore no longer meets it, so that the arc AB has not properly speaking a trigonometrical tangent. We say, indeed, that the tangent of AB or 90° is *infinite*; by this expression, however, we mean, that if the difference between an arc and 90° is indefinitely small, the tangent of this arc will be indefinitely great, that is, greater than any assignable quantity. From B to D the tangents decrease, until at D the tangent is 0; from this point they increase, until at E the tangent is again infinite; from E they decrease, and the tangent becomes 0 at A.

With respect to the changes, which occur in the positions of the tangents, from A to B the radius CF produced meets the indefinite tangent at A above the diameter AD; in the first quadrant therefore we consider the tangents *positive*. From the point B to D, the radius F′C produced no longer meets the indefinite tangent at A above the diameter AD, but below it; in the second quadrant then the tangents are *negative*. Tracing in like manner the changes of position, which take place with respect to the tangents in the remaining quadrants, we find, that in the third quadrant they again become *positive*, and in the fourth *negative*.

97. We have traced the changes, which occur with respect to the sines, cosines, &c. from the departure of the point F from A until its return to this point again. We may now suppose that at A it commences a second revolution, and regarding as one arc the whole course passed over by the point F from the commencement of its motion, we shall have arcs, that exceed a circumferance, and which have the same sines, cosines, tangents &c. as those described in the first revolution.

98. The formulas for the sine and cosine of the *sum*, and the *difference* of two arcs obtained by a geometrical demonstration art. 35, 36 are true whatever the magnitude of these arcs. Representing the arcs by the letters a and b respectively, those formulas, radius being considered as unity, may now be expressed in algebraic language as follows

1. $\sin (a + b = \sin a \cos b + \sin b \cos a$
2. $\sin (a - b) = \sin a \cos b - \sin b \cos a$
3. $\cos (a + b) = \cos a \cos b - \sin a \sin b$
4. $\cos (a - b) = \cos a \cos b + \sin a \sin b$

99. The four preceding formulas form the basis of trigonometrical analysis. We shall now employ them in the preparation of various other formulas of the greatest utility in the more extended application of the trigonometrical lines. Before making use of them, however, for this purpose, it may be well, in order to see the precise agreement between the analytic and geometrical mode of considering the subject, to show from these expressions the variations, both in sign and magnitude, of the sines and cosines through the first circumference.

Let p denote a semicircumference; then the cosine of $\frac{1}{2} p$ is 0, and its sine is 1; hence substituting $\frac{1}{2} p$ for a in the preceding formulas, we have

$\sin (\frac{1}{2} p \pm b) = \cos b, \quad \cos (\frac{1}{2} p \pm b) = \mp \sin b.$

To show the absolute value of these expressions, let the arc BF′ (fig. 34) $= b$; since the arc AB $= \frac{1}{2} p$, the arc AF′ $= (\frac{1}{2} p + b)$; then G′F′ the sine of AF′, and also of DF′, is of consequence the cosine of BF′; hence $\sin (\frac{1}{2} p + b) = \cos b$; CG′ moreover the cosine of AF′ is the sine of BF′; hence $\cos (\frac{1}{2} p + b) = \sin b$. If we make BF $= b$, then $\sin (\frac{1}{2} p - b) = \cos b$, and $\cos (\frac{1}{2} p - b) = \sin b$.

With respect to the sign —, which affects the absolute value of $\cos (\frac{1}{2} p + b)$, it signifies, that if we regard as positive the sine and cosine of an arc less than a quadrant, the cosine of an arc greater than a quadrant, but less than a semicircumference will be negative, while its sine is positive.

If we make $b =$ a quadrant, then $\sin (\frac{1}{2} p \pm b) = 0$, and $\cos (\frac{1}{2} p \pm b) = \mp 1$. Again let $a = p$, then $\sin (p \pm b) = \mp \sin b$, and $\cos (p \pm b) = - \cos b$; if we make $b = \frac{1}{2} p$, then $\sin \frac{3}{2} p = - 1$, and $\cos \frac{3}{2} p = 0$. Lastly let $a = \frac{3}{2} p$, then $\sin (\frac{3}{2} p \pm b) = - \cos b$, and $\cos (\frac{3}{2} p \pm b) = \pm \sin b$. The absolute values of these last expressions may also be easily verified; the signs show that every arc comprehended between p and $\frac{3}{2} p$ has its sine and cosine negative, while an arc comprehended between $\frac{3}{2} p$ and $2 p$ has its sine negative, but its cosine positive.

Expressions for the sines and cosines of multiple arcs.

100. Putting in the expression for $\sin (a + b)$ no. 1 art. 98, $b = a$, $2 a$, $3 a$, &c, we shall have

$\sin 2 a = 2 \sin a \cos a$

$\sin 3 a = \sin a \cos 2 a + \sin 2 a \cos a$

$\sin 4 a = \sin a \cos 3 a + \sin 3 a \cos a$

$\sin 5 a =$ &c

Making the same substitutions in the expression for cos $(a+b)$ no. 3 art. 98, we have

$$\cos 2a = \cos^2 a - \sin^2 a$$
$$\cos 3a = \cos a \cos 2a - \sin a \sin 2a$$
$$\cos 4a = \cos a \cos 3a - \sin a \sin 3a$$
$$\cos 5a = \&c.$$

By the expressions $\sin^2 a$, $\cos^2 a$, &c, we understand the square of the sine of a, square of the sosine of a, &c.

101. To find expressions for $\sin 3a$, $\sin 4a$, $\cos 3a$, $\cos 4a$ &c in terms of the simple arc, we substitute in the formulas just obtained the values of $\sin 2a$, $\cos 2a$ &c. Thus we have

$$\sin 3a = 3 \sin a \cos^2 a - \sin^3 a$$
$$\sin 4a = 4 \sin a \cos^3 a - 4 \sin^3 a \cos a$$
$$\sin 5a = \&c.$$
$$\cos 3a = \cos^3 a - 3 \sin^2 a \cos a$$
$$\cos 4a = \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a$$
$$\cos 5a = \&c.$$

Expressions for the sines and cosines of half a given arc

102. In the expresssion for $\cos 2a$ art. 100, making $a = \frac{1}{2}a$, we have $\cos^2 \frac{1}{2}a - \sin^2 \frac{1}{2}a = \cos a$ (1)

but art. 30 $\cos^2 \frac{1}{2}a + \sin^2 \frac{1}{2}a = 1$ (2)

whence by subtraction $2 \sin^2 \frac{1}{2}a = 1 - \cos a$

wherefore $\sin \frac{1}{2}a = (\frac{1}{2} - \frac{1}{2}\cos a)^{\frac{1}{2}}$

Adding next the equations 1 and 2, and proceeding in like manner, we have $\cos \frac{1}{2}a = (\frac{1}{2} + \frac{1}{2}\cos a)^{\frac{1}{2}}$

Expressions for the products of sines and cosines.

103. Adding the equations 1 and 2 art. 98, we have

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$$

whence $\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$ (1)

Subtracting the second of the same equations from the first and reducing, we have

$$\sin b \cos a = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b) \quad (2)$$

In like manner the equations, 3 and 4 art. 98 give

$$\cos a \cos b = \tfrac{1}{2} \cos (a + b) + \tfrac{1}{2} \cos (a - b) \qquad (3)$$

$$\sin a \sin b = \tfrac{1}{2} \cos (a - b) - \tfrac{1}{2} \cos (a + b) \qquad (4)$$

Expressions for the powers of sines and cosines.

104. 1. Let $b = a$ in nos 3 and 4 of the preceding article, and we have

$$\sin^2 a = \tfrac{1}{2} - \tfrac{1}{2} \cos 2a$$

$$\cos^2 a = \tfrac{1}{2} + \tfrac{1}{2} \cos 2a$$

2. Multiplying the first of the preceding equations by $\sin a$, or to avoid fractions by $4 \sin a$, we have

$$4 \sin^3 a = 2 \sin a - 2 \sin a \cos 2a$$

but art. 103 $\quad \sin (a + b) - \sin (a - b) = 2 \sin b \cos a$

in this last equation let $a = 2a$ and $b = a$, then

$$\sin 3a - \sin a = 2 \sin a \cos 2a$$

wherefore by substitution and reduction

$$\sin^3 a = \frac{3 \sin a - \sin 3a}{4}$$

and by a process altogether similar, we obtain

$$\cos^3 a = \frac{3 \cos a + \cos 3a}{4}$$

3. In like manner, we find

$$\sin^4 a = \frac{\cos 4a - 4 \cos 2a + 3}{8}$$

$$\cos^4 a = \frac{\cos 4a + 4 \cos 2a + 3}{8}$$

Expressions for the tangents of arcs in terms of the tangents.

105. By art. 48, we have $\tan (a + b) = \dfrac{\sin (a + b)}{\cos (a + b)}$,

whence by substitution $\tan (a + b) = \dfrac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$

dividing the numerator and denominator of the second member of this equation by cos a cos b, we have for the numerator

$$\frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b} = \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} = \tan a + \tan b$$

and for the denominator

$$\frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b} = 1 - \frac{\sin a}{\cos a} \times \frac{\sin b}{\cos b} = 1 - \tan a \tan b$$

whence by substitution and reduction, we obtain

$$\tan (a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \quad (1)$$

In like manner we obtain $\tan (a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$ (2)

By art. 57, we have $\cot (a + b) = \frac{1}{\tan (a + b)}$, whence by substitution

$$\cot (a + b) = \frac{1 - \tan a \tan b}{\tan a + \tan b} \quad (3)$$

So also

$$\cot (a - b) = \frac{1 + \tan a \tan b}{\tan a - \tan b} \quad (4)$$

Expressions for the tangents and cotangents of multiple arcs in terms of the simple arcs.

106. Putting successively $b = a, 2a, 3a$,&c, in nos 1 and 3 of the preceeding article and substituting for factors of multiple arcs, we have

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

$$\tan 4a = \frac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a}$$

$$\tan 5a = \&c.$$

The expressions for the cotangents, will it is easy to see, be merely an inversion of the preceding.

Expressions for the tangents and cotangents of half an arc.

107. By art. 48 we have $\tan \frac{1}{2} a = \frac{\sin \frac{1}{2} a}{\cos \frac{1}{2} a}$, whence by substitution art. 102

$$\tan \tfrac{1}{2} a = \frac{\frac{(1 - \cos a)^{\frac{1}{2}}}{\sqrt{2}}}{\frac{(1 + \cos a)^{\frac{1}{2}}}{\sqrt{2}}} = \frac{(1 - \cos a)^{\frac{1}{2}}}{(1 + \cos a)^{\frac{1}{2}}}$$

multiplying numerator and denominator by $(1 + \cos a)^{\frac{1}{2}}$

$$\tan \tfrac{1}{2} a = \frac{(1 - \cos^2 a)^{\frac{1}{2}}}{1 + \cos a} = \frac{\sin a}{1 + \cos a}$$

The expression for the cotangent $\frac{1}{2} a$, it is evident, will be the inverse of that for the tangent $\frac{1}{2} a$.

Expressions for secants and cosecants.

108. By art. 48 we have $\sec (a + b) = \frac{1}{\cos (a + b)}$

whence by substitution

$$\sec (a + b) = \frac{1}{\cos a \cos b - \sin a \sin b}$$

dividing both numerator and denominator by $\cos a \cos b$ and substituting we have

$$\sec (a + b) = \frac{\sec a \sec b}{1 - \tan a \tan b}$$

so also

$$\sec (a - b) = \frac{\sec a \sec b}{1 + \tan a \tan b}$$

And by a process altogether similar we obtain

$$\operatorname{cosec} (a \pm b) = \frac{\sec a \sec b}{\tan a \pm \tan b}$$

Making $b = a$ in the expressions for sec and cosec $a + b$, we have

$$\sec 2 a = \frac{\sec^2 a}{1 - \tan^2 a}$$

$$\operatorname{cosec} 2 a = \frac{\sec^2 a}{2 \tan a}$$

109. From what has been done, it will be perceived, that an almost infinite variety of trigonometrical formulas may be made adapted to the purposes, for which they may be required; we shall close the subject, however, with the investigation of formulas for finding the angle of a triangle when the sides are given, and also for determining the area of a triangle from different data.

110. In the triangle ABC (fig. 33), let fall from one of the angles, B for example, upon the side AC the perpendicular BD; then (Geom. 191, 192) $AB^2 = AC^2 + BC^2 \mp 2\,AC \times DC$, the sign — being used when the perpendicular falls within the triangle, and the sign + when it falls without; but in the right angled triangle BDC, radius being 1, $DC = BC \times \cos C$, wherefore $AB^2 = AC^2 + BC^2 - 2\,AC \times BC \times \cos C$, the sign — being sufficient for the term 2 AC &c. because when the angle C is obtuse its cosine is negative, and consequently changes — into + as is required by the geometrical construction. Hence, employing the letters a, b, c, to represent the sides BC, AC, AB, and deducing the value of cos C, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$$

but this expression not being well adapted to calculation by logarithms, another is to be sought.

The formula $\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos 2\,a$ gives by substituting $\frac{1}{2}$ C for a, and deducing the value of cos C,

$$\cos C = 1 - 2\sin^2 \tfrac{1}{2} C$$

comparing the two values of cos C, and deducing that of $2\sin^2 \frac{1}{2} C$, we have

$$2\sin^2 \tfrac{1}{2} C = 1 + \frac{c^2 - a^2 - b^2}{2\,ab} = \frac{c^2 - a^2 - b^2 + 2\,ab}{2\,ab}$$

$$= \frac{c^2 - (a - b)^2}{2\,ab} = \frac{(c + a - b)\,(c - a + b)}{2\,ab}$$

whence

$$\sin^2 \tfrac{1}{2}C = \frac{(c+a-b)(c-a+b)}{4\,ab} = \frac{\frac{1}{2}(c+a-b)\frac{1}{2}(c-a+b)}{ab}$$

but $\frac{1}{2}(c+a-b) = \frac{1}{2}(c+a+b) - b$, and $\frac{1}{2}(c-a+b) = \frac{1}{2}(c+a+b) - a$, whence representing $c+a+b$ by s and reducing, we have

$$\sin \tfrac{1}{2}C = \left(\frac{(\frac{1}{2}s-a)(\frac{1}{2}s-b)}{ab}\right)^{\frac{1}{2}}$$

deducing next the value of cos C from the formula $\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos 2\,a$, we obtain by a process altogether similar

$$\cos \tfrac{1}{2}C = \left(\frac{\frac{1}{2}s\,(\frac{1}{2}s-c)}{ab}\right)^{\frac{1}{2}}$$

Expressions for the surfaces of triangles.

1. Given two angles and the included side to find the surface of the triangle.

Let AC (fig. 33) be the given side A and C the given angles; employing the same notation as above, we have

$$c = \frac{b \sin C}{\sin B} = \frac{b \sin C}{\sin (A+C)}$$

the sine of an angle being the same with that of its supplement. Calling d the perpendicular let fall upon the given side, we have $d = c \sin A$, whence

$$d = \frac{b \sin A \sin C}{\sin (A+C)}$$

multiplying this expression by $\frac{1}{2}b$ in order to obtain the surface sought, and denoting this surface by S, we have

$$S = \frac{b^2 \sin A \sin C}{2 \sin (A+C)}$$

Substituting next for sin (A + C) its value art. 98, and dividing both numerator and denominator by sin A sin C, we have finally

$$S = \frac{b^2}{2\,(\cot A + \cot C)}$$

2. Given two sides and the included angle to find the surface of the triangle.

Let a and b be the given sides (fig. 33), C the given angle, we have $d = a \sin C$; whence multiplying the perpendicular by $\frac{1}{2} b$ we have

$$S = \tfrac{1}{2} ab \sin C$$

3. Given the three sides to find the surface of the triangle.

In the triangle ABC (fig. 33) we have $DC = a \cos C$, designating BD by d as before, we have by the known property of a right angled triangle.

$$d^2 = a^2 - a^2 \cos^2 C$$

Substituting for cos C its value in art. 110 we have

$$d^2 = a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2$$

whence $\qquad 4 b^2 d^2 = 4 a^2 b^2 - (a^2 + b^2 - c^2)^2$

but the two terms of the right hand member being the difference between two squares, we have

$$4 b^2 d^2 = (2 ab + a^2 + b^2 - c^2) (2 ab - a^2 - b^2 + c^2)$$
$$= ((a + b)^2 - c^2) (c^2 - (a - b)^2)$$

the two factors of this last being each also the difference between two squares, we have

$$4 b^2 d^2 = (a + b + c) (a + b - c) (c + a - b) (c + b - a)$$

adding and subtracting c, b and a in the second third and last factor respectively, and designating $a + b + c$ by $2 s$, we have

$$4 b^2 d^2 = 2 s (2 s - 2 c) (2 s - 2 b) (2 s - 2 a)$$
$$= 16 s (s - c) (s - b) (s - a)$$

dividing both sides by 16 and extracting the square root

$$\frac{bd}{2} = (s (s - c) (s - b) (s - a))^{\frac{1}{2}}$$

Hence, to find the surface of a triangle by means of the three sides, *we must subtract successively each side from the half* *sum, multiply the half sum and the three remainders continually together, and take the square root of this product.*

MISCELLANEOUS PROBLEMS.

1. A flag-staff is placed upon a wall 40 feet long in such a situation, that a line of 34 feet in length will reach from its top to one end of the wall and a line of 25 feet from its top to the other end. What is the height of the flag-staff, and its distance from the ends of the wall ?

2. From the top of a tower, whose height is known, was observed the angular distance between two objects situated in the same horizontal plane with its base,and also the angles of depression of each of them. To determine from these data the distance between the two objects, the objects not being in the same horizontal plane.

3. Let A and C be two stations on a sloping ground distant 410 yards, O an object on the top of a hill ; the following angles being found by observation, viz. OCA 79° 29′, OAC 63° 11′, and the angles of elevation at A and C 6° 36′ and 5° 22′ respectively ; it is required to calculate the height and distance of the object from each station.

4. At the same instant two observers take the elevation of a balloon D at two stations A and B, and also the angles DAB, DBA ; the distance AB is m yards and the elevation of B above A is n° ; to determine from these data the elevation of the balloon and its distance from the lower station A.

5. The distances between three points A, B and C (fig.48) are given, and also at a point D in the line BA produced the angle ADC is found by observation to be n°, to determine the distance from the point D to each of the points A, B and C.

6. From the top and bottom of a building, whose height is known, the elevation of another building on the same hori-

zontal plane is observed, to find that point in the plane, which is equidistant from the top of each.

7. How must three trees A, B, C, be planted, so that the angle at A may be double the angle at B, and the angle at B double the angle at C and a line of 400 yards may just go round them?

8. The distances between three points A, B, C (fig. 48) being known, it is required to find the position of the point D with reference to these points, the angles ADB, ADC being found by observation equal to m° and n° respectively.

Let the given distances AB, AC, BC be represented by the letters a, b, c, respectively; in the triangle ADB let the angle ABD be represented by x, and in the triangle ADC the angle ACD by y; the triangles ADB ADC give the equation.

$$a \sin n \sin x = b \sin m \sin y$$

but in the quadrilateral ABDC we have

$$y = 4 \text{ right angles} - \text{ADB} - \text{ADC} - \text{BAC} - x.$$

but since the angle BAC is known when the three sides of the triangle ABC are given, the first four terms of the right hand member of this last equation are known; representing these by d for the sake of conciseness, and substituting for y its value in the first equation, we have

$a \sin n \sin x = b \sin m \sin (d - x) = b \sin m (\sin d \cos x - \sin x \cos d)$, dividing both sides by $\sin x$ we have

$$a \sin n = b \sin m (\sin d \cot x - \cos d)$$

whence

$$\cot x = \frac{a \sin n + b \sin m \cos d}{b \sin m \sin d}$$

9. A person travelling along a road BA (figure 49) observes at A the elevation of a tower CD, the perpendicular distance of which BD from the road is known. At the same time he also observes the angular distance CAB of the top of the tower from an object B in the road. To determine from these data the height of the tower.

In the triangle CAD we have R : AC : : sin CAD : CD
and in the triangle CAB R : AC : : sin CAB : CB
whence sin CAD : sin CAB : : CD : CB
but in the triangle CBD, we have CD : CB : : sin CBD : R
comparing this last with the preceding, we obtain R being 1

$$\sin CBD = \frac{\sin CAD}{\sin CAB}$$

the value of CBD being determined, the height CD is readily found.

10. At a station in a right line joining the bases of two mountain peaks I observed the angle of elevation of each of the peaks ; proceeding thence 100 yards further on level ground in a direction perpendicular to the line joining the bases of the peaks, I again observed their angles of elevation. How may the distance between the two peaks be determined from these data.

11. From a window A in the side of a tower exactly 50 feet from the ground, the angle of depression of the bottom of a may-pole was found to be $m°$, it was also observed that the angle of elevation of the top of the pole at this station was equal to the complement of its angle of elevation at the bottom of the tower ; to determine from these data the height of the tower, (fig. 51).

By the question the side AB and the angle BAD are given, and the angle EAC is the complement of ABC ; the angle ACD is equal to 90° — ABC, and the angle ACB to 90°— 2 ABC.

In the triangle ACB we have sin ACB : AB : : sin ABC: AC
and in the triangle ACD sin ACD : AD : : sin ADC: AC
whence

$$\frac{AD \sin ADC}{\sin (90° - ABC)} = \frac{AB \sin ABC}{\sin (90° - 2\,ABC)}$$

or

$$\frac{AD \sin ADC}{\cos ABC} = \frac{AB \sin ABC}{\cos 2\,ABC}$$

from which we have art. 37, 48, $\tan 2\,ABC = \frac{2\,AD \sin ADC}{AB}$ AD being found by means of the triangle ABD, we have by the above formula the angle ABC, and this being found the height sought will readily be determined.

12. Having at a certain unknown distance taken the angle of elevation DAB of a steeple DE (fig. 44) I advanced to a point B 60 yards nearer on level ground and there observed the angle of elevation to be the complement of the former, advancing still nearer 20 yards to another point C, the angle of elevation there appeared to be just double of the first. To determine the altitude of the steeple.

Putting $DAB = x$, we have by the question $DBC = 90° - x$, and $DCE = 2x$; whence $BDC = 3x - 90°$, and $ADB = 90° - 2x$

in the triangle ADB we have $\sin(90° - 2x) : 60 :: \sin x : DB$
and in the triangle BDC $\sin(3x - 90°) : 20 :: \sin 2x : DB$
whence

$$\frac{20 \sin 2x}{\sin(3x - 90°)} = \frac{60 \sin x}{\sin(90° - 2x)} \qquad (1)$$

but the triangle ACD being isosceles, we have $DC = 80$ yards, whence in the triangle BDC we have art. 69
$100 : 60 :: \sin(90° - x) + \sin(3x - 90°) :: \sin(90° - x) - \sin(3x - 90°)$, from which we obtain $\sin(90° - x) = 4 \sin(3x - 90°)$ whence by substitution in equation (1)

$$\frac{20 \sin 2x}{\frac{1}{4} \sin(90° - x)} = \frac{60 \sin x}{\sin(90° - 2x)}$$

substituting next for $\sin 2x$ its value in terms of x, and for the sines of the angles in the denominators the cosines of their complements, we obtain finally $\cos 2x = .37500$, and the angle $2x$ or DCE being thus obtained the height DE will be easily found.

13. A tower whose height is 60 feet is situated on the side of a hill, which rises due N. at an angle BDE of 20°;

what length of ladder will be necessary to scale the tower, supposing the foot of the ladder to be placed at a distance BC of 30 feet in a direction S 40° W from the base of the tower, (fig. 52).

The lines BD and BC being referred to a horizontal plane, we have by the question the horizontal angle CED = 40° ; CD moreover being supposed perpendicular to DE, the triangle CDE will be right angled at D.

From the triangle BCE we obtain radius being 1, BE = 30 sin BCE, and from the triangles BCE, DCE, DE = 30 sin 50° cos BCE.

In the triangle BDE we have

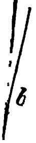
$\mathrm{DE} + \mathrm{BE} : \mathrm{DE} - \mathrm{BE} :: \sin 70° + \sin 20° : \sin 70° - \sin 20°$

putting for the sake of conciseness the letters a and b for the last two terms of this proportion respectively, and substituting for DE and BE the values found above, we have

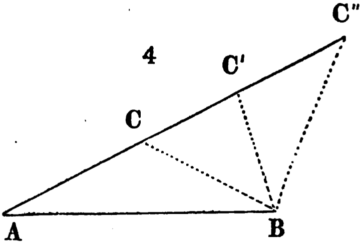
$$(a + b) \sin \mathrm{BCE} = (a - b) \sin 50° \cos \mathrm{BCE}$$

whence
$$\tan \mathrm{BCE} = \frac{(a - b) \sin 50°}{(a + b)}$$

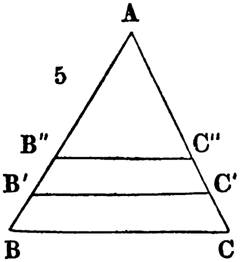
and BCE being determined by this formula the length sought will be readily found.

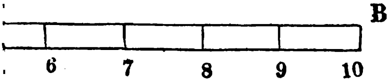


4

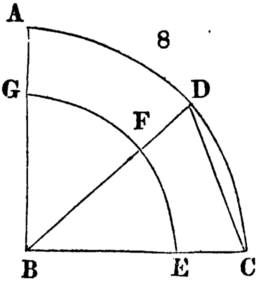


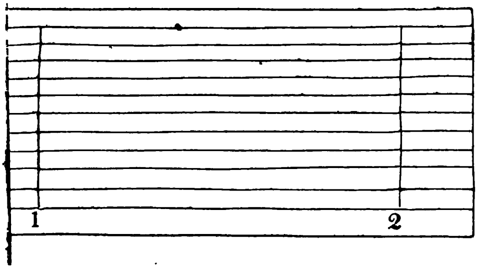
5



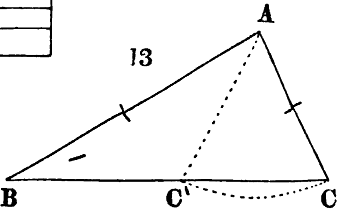


8

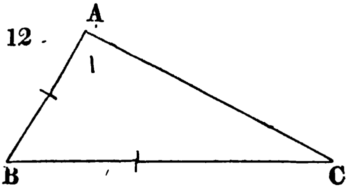




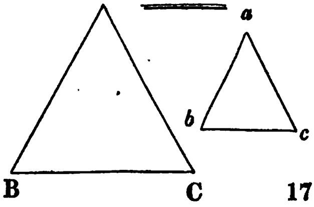
13



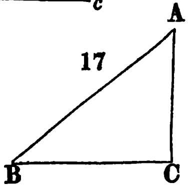
12



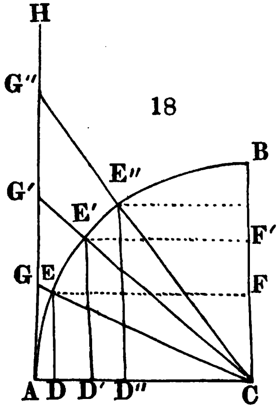
15



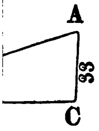
17



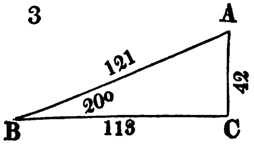
18



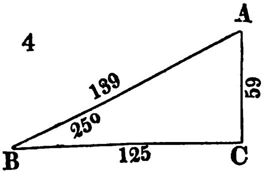
L. T. JACKSON.

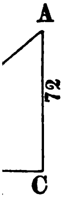


3

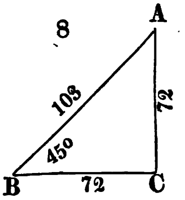


4

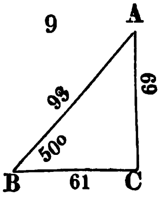




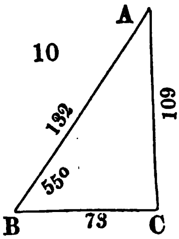
8



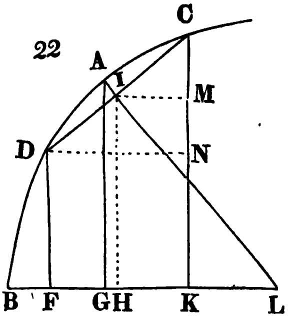
9



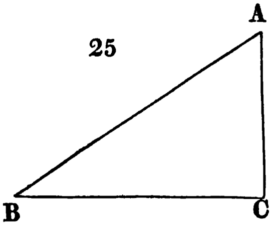
10



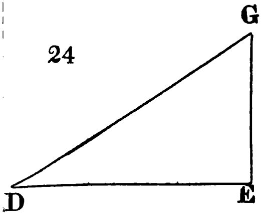
22



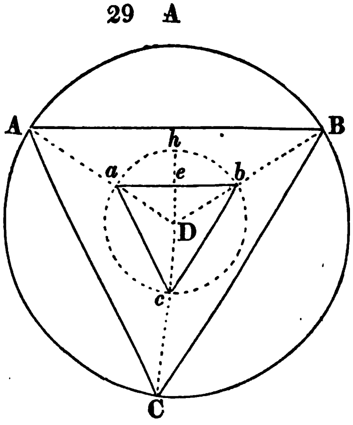
25



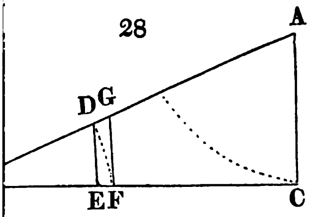
24



29

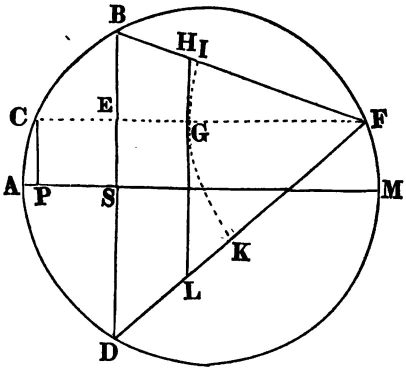


28

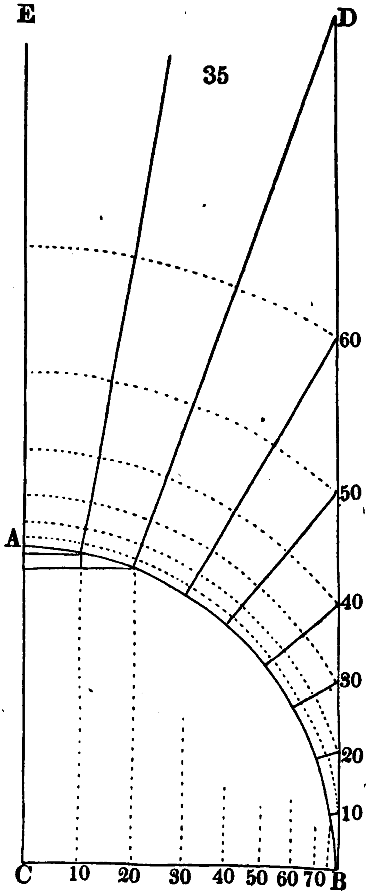


L. T. JACKSON.

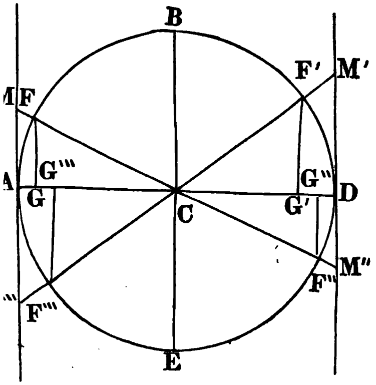
31



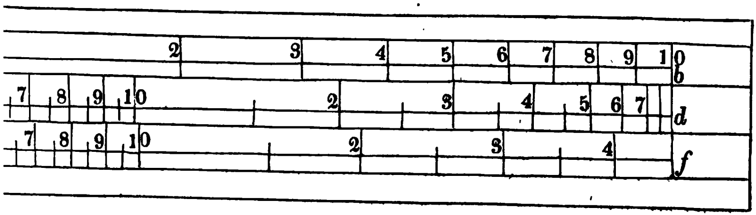
35



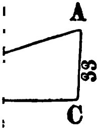
34



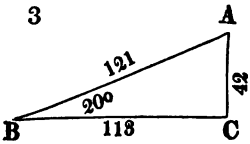
36



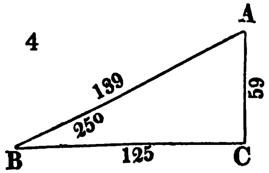
L. T. JACKSON.

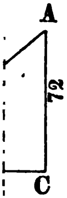


3

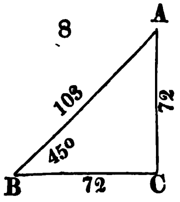


4

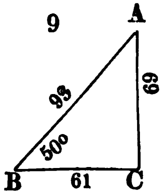




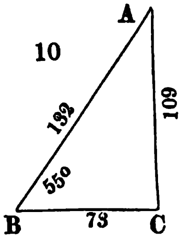
8



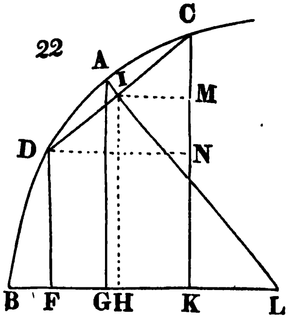
9



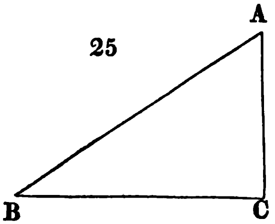
10



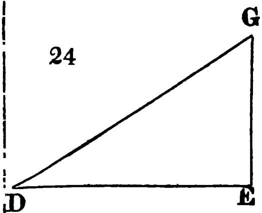
22



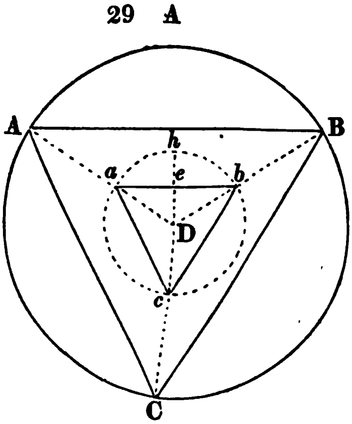
25



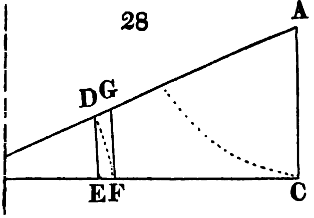
24



29

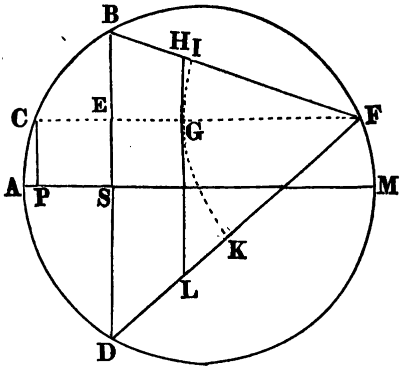


28

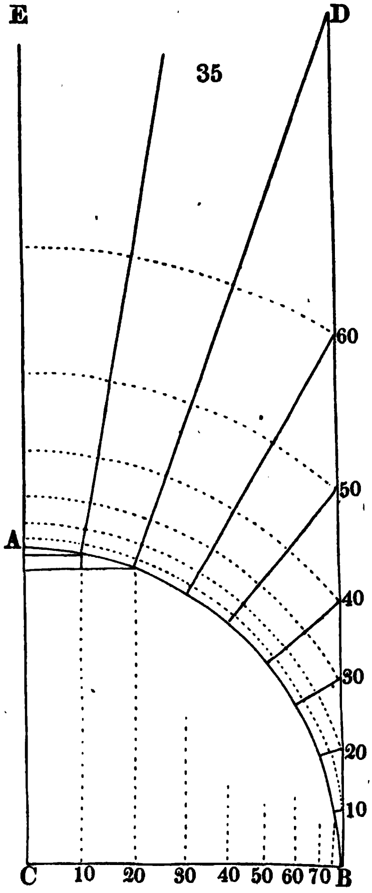


L. T. JACKSON.

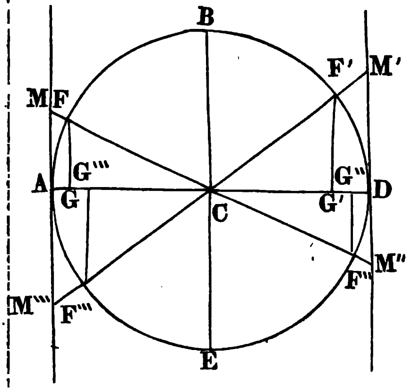
31



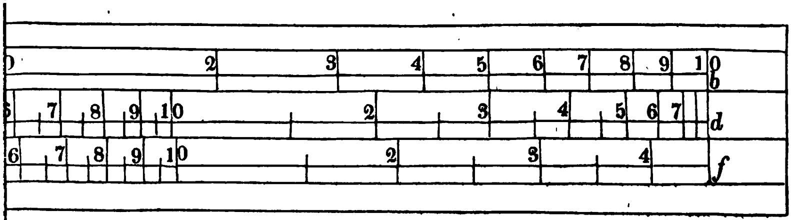
35



34

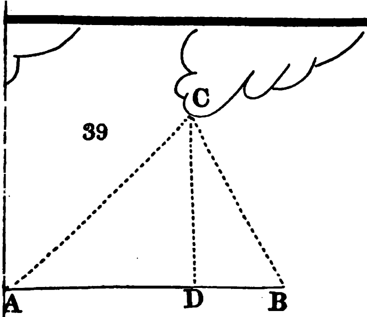


36

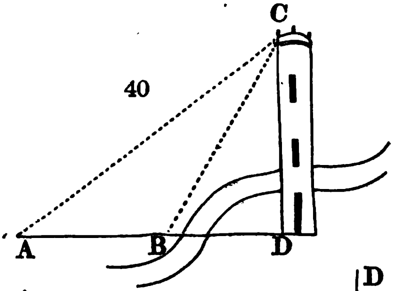


L. T. JACKSON.

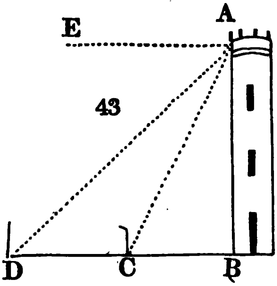
39



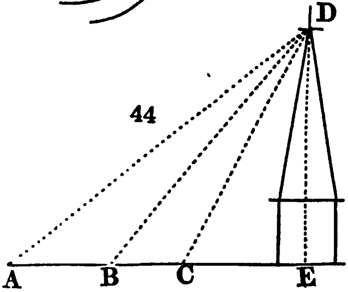
40



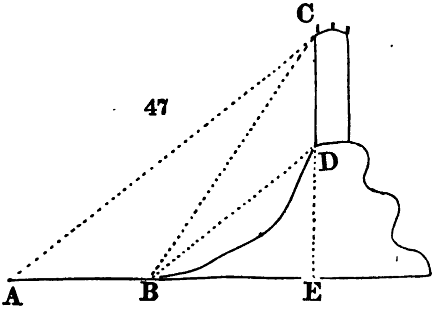
43



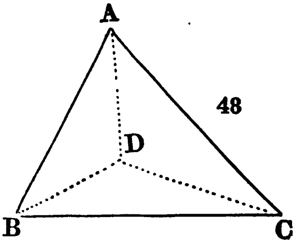
44



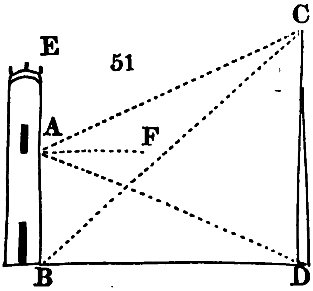
47



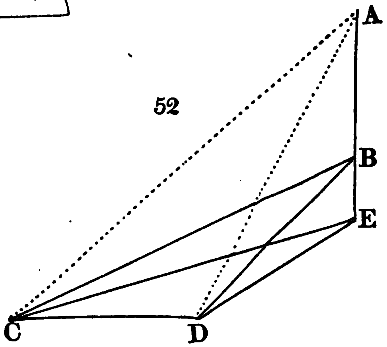
48



51



52



L. T. JACKSON.